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Spectral and ergodic properties of completely positive maps

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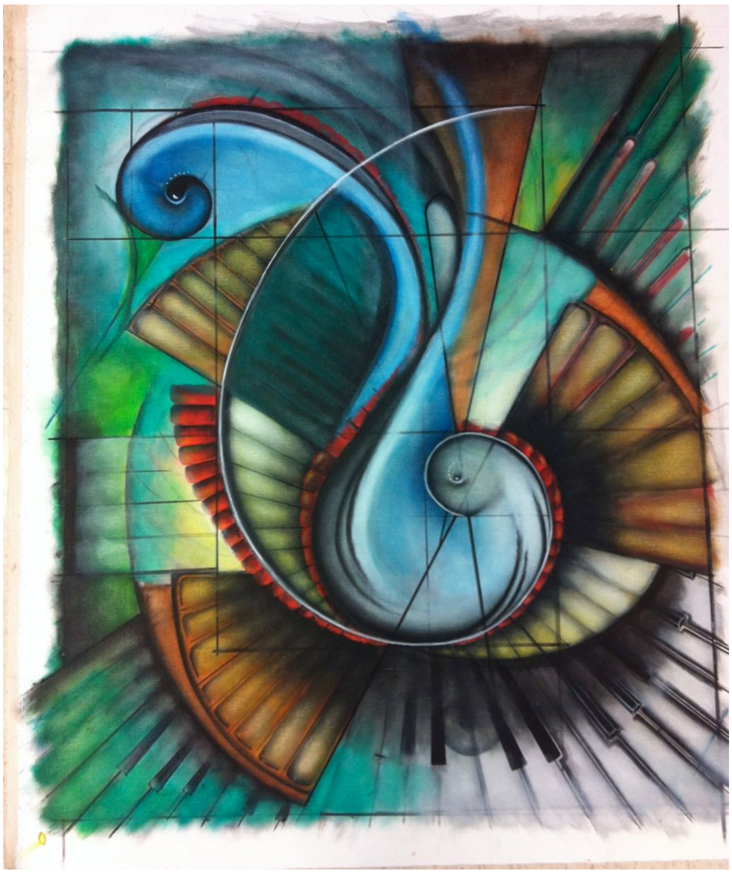

Figure 1: *"Fibonacci"*, by Bruno Hernani.

Abstract

Given any discrete C^*-dynamical system, by which we mean a pair $(\mathfrak{A}, \Phi)$, where $\mathfrak{A}$ is a (possibly non-commutative) unital C^*-algebra and Φ a unital completely positive map, there is a circle of natural problems we woud like to highlight here because they are the main motivation underlying the present work.

1. The first problem has to do with separating the transient part of the dynamics from its persistent part. More precisely, this means being able to decompose the C^*-algebra $\mathfrak{A}$ into a direct sum of two components, say $\mathfrak{A} = \mathfrak{A}_\infty \oplus \mathfrak{A}_0$, which at this level are not necessarily C^*-algebras, such that for any $a \in \mathfrak{A}_\infty$ one has $\|\Phi(a)\| = \|a\|$, while for any $a \in \mathfrak{A}_0$ one has $\lim_{n\to\infty} \|\Phi^n(a)\| = 0$. Intuitively, $\mathfrak{A}_\infty$ comes from the peripheral spectrum of Φ, that is to say $\sigma_{ph}(\Phi) := \sigma(\Phi) \cap \mathbb{T}$, which is never empty since $1 \in \sigma_{ph}(\Phi)$. On the contrary, $\mathfrak{A}_0$ is associated with the rest of the spectrum. In general, this is a rather hard technical problem to tackle. Even so, we do treat it successfully if we also assume that the spectrum of Φ displays what is called a mass gap, which is certainly the case when $\mathfrak{A}$ is finite dimensional. This also poses the problem of studying what the spectrum of such a map looks like in general. To the best of our knowledge, a satisfactory answer is not available even for endomorphisms.

2. The second problem to deal with is whether the persistent part of the system, namely $\mathfrak{A}_\infty$, which always carries an operator system structure, also has an intrinsic C^*-algebra structure up to redefining the product.

3. Lastly, the final question to answer is whether the given map Φ restricts to the persistent part as a *-automorphism with respect to the new product which makes it into a C^*-algebra.

Introduction

It is quite a remarkable fact that a satisfactory unified picture of both classical and quantum mechanics can be given in the framework of C^*-*algebra theory*.
What this brief introduction aims to do is provide a rather quick account of how this can be done. As is known, the evolution of a classical system can always be framed in the following formalism. An evolution or *dynamics* on a measurable space X, i.e. a set equipped with a $\sigma-$algebra $\mathcal{F}$, is nothing but a family of maps $\varphi_t : X \to X$ indexed by time t running on $\mathbb{R}$, such that $(x, t) \mapsto \varphi_t(x)$ is measurable and the group property $\varphi_t \circ \varphi_s = \varphi_{t+s}$ holds.
From a more physical perspective, the map φ_t encodes the evolution. Phrased differently, if $x_0 \in X$ represents the state of our system at the initial time, say $t = 0$, then $\varphi_t(x_0) \in X$ will represent its state at time t.
There are, however, some situations when it may be more convenient to consider discrete evolution, as in the following motivating example by [7]:

Example 0.0.1. Suppose to have an (ideal) gas contained in a box represented by d moving particles. Each of them is described by six coordinates (three for position, three for velocity). Then the situation of the gas (or *state*) of the system, is given by a point $x \in \mathbb{R}^{6d}$. Clearly not all points in $\mathbb{R}^{6d}$ can attained by our gas, so we restrict our attention to the set X of all possible states and call this the *state space* of the system.
Also, the system changes while time is running, and therefore a given state (a point in X) also "moves" in X. This movement is governed by Newton's laws of mechanics and then by Hamilton's differential equations. The solutions to these equations determine a map $\varphi : X \to X$ in the following way: if at time $t = 0$, our system is in the state $x_0 \in X$, then at time $t = 1$ it will be in a new state x_1, and we define φ by $\varphi(x_0) := x_1$. As a consequence, at time $t = 2$ the state x_0 becomes

$$x_2 := \varphi(x_1) = \varphi^2(x_0)$$

and

$$x_n := \varphi^n(x_0).$$

for time t running in $\mathbb{N}$.
In the thermodynamic picture, generally d is a very large number, of the order $\sim 10^{23}$, which makes it practically impossible to solve the motion equations in totality. But in general, we are interested in the average values of the quantities, and this leads to the study of so-called *DLR states (Dobrushin Lanford-Ruelle)*. In a similar fashion, in the *quantum case*, we are interested in the study of so-called *KMS states (Kubo Martin Schwinger)* (see [9] and [20]).

The above example seems to suggest the following definition.

Definition 0.0.1. *A pair (X, φ), consisting of a state space X and a (usually invertible) map $\varphi : X \to X$ is called a discrete classical dynamical system.*

Given $x \in X$, the set $\mathfrak{O}(x) := \{\varphi_t(x) : t \in \mathbb{R}\}$ is thus the *orbit* of x under the dynamics prescribed by φ_t. Obviously, if the dynamics is discrete, then the orbit must be defined as $\mathfrak{O}(x) := \{\varphi^n(x) : n \in \mathbb{Z}\}$.
Now what we are interested in is not quite to delve into the properties of the dynamics itself. In fact, our primary concern here is to explain the structure of the so-called *observables* of our system that emerges from this description.
From a physical standpoint, the observables are just those measurable quantities associated with the state of our system. In mathematicians' parlance, therefore, they should be construed as real functions $f : X \to \mathbb{R}$ assigning to each $x \in X$ the value $f(x)$ of a specific measurement, for instance the temperature.
If we denote by $\mathcal{O} := \mathcal{C}(X, \mathbb{R})$ the set of observables of a dynamical system, it is possible to turn the set of observables into a commutative algebra:

Definition 0.0.2. *For $x \in X$, $f, g \in \mathcal{O}$ and $a \in \mathbb{R}$ one can define*

- $(f + g)(x) := f(x) + g(x)$,
- $(af)(x) := af(x)$,
- $(fg)(x) := f(x)g(x)$,
- $\|f\| = sup\{|f(x)| : x \in X\}$.

Furthermore, as soon as our set X is also endowed with a locally compact Hausdorff topological space structure we are led to the following description.

Definition 0.0.3. *The set of observables $\mathcal{O}$ of a classical dynamical system corresponds exactly to the set of self-adjoint elements of a commutative, in general non-unital C^*-algebra $\mathfrak{A}$.*

The definition given above is so general as to include Hamiltonian mechanics, as shown in the next example

Example 0.0.2. In Hamiltonian mechanics the algebra of observables is given by the functions on the phase space that vanish at infinity, say $C_0(X)$, and the phase space itself X also carries a differential structure. More precisely, it is a symplectic manifold of dimension $2n$, where n is the number of degrees of freedom of the system under consideration. For instance, $n = 3$ for a free point particle moving in the three-dimensional space.
Furthermore, having a unital algebra corresponds to the physical situation where our system is bound in a compact region M of X, which is more exactly a compact sub-manifold. Tipically, compact submanifolds of this sort are obtained as the level sets of suitable prime integrals, such as the energy of the system as long as it is conservative.
As for the time evolution, Hamiltonian mechanics is entirely deterministic. In other terms, if the state x_0 of our system is known at the inital time $t = 0$, then in principle the state x_t at any future (or past) time t is also known and it is determined the Hamiltonian flow,

i.e. the one-parameter group $\{\Phi_t : t \in \mathbb{R}\}$ which encodes the evolution in such a way that $x_t = \Phi_t(x_0)$, is obtained by solving the Hamilton equations. As is known, in local charts the Hamilton equations take the simple form

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$$

where H is the so-called Hamiltonian of the system, namely a smooth function $H : X \times \mathbb{R} \to \mathbb{R}$, which in general might depend on time t also. Furthermore, in most situations it just represents the total energy of the system and is preserved by the dynamics provided that it does not depend explicity on time, that is if $\frac{\partial H}{\partial t} = 0$

Once we have clarified what the matematical structure of the observables should be, we can move on to describe what it should be meant in general by a state of the system. First, the definition should be so general as to include the points of X itself, which should be interpreted as the so-called pure states, namely those states for which the information we have on the system is as much as possible.
Second, physical reasons suggest that the set of states should also have a convex structure. A rather felicitous way to combine these two requirements together is to define states in terms of linear functionals on $\mathcal{O}$:

Definition 0.0.4. *A state on $\mathcal{O}$ is a positive bounded linear functional on $C(X)$, i.e. a linear functional $\varphi : C(X) \to \mathbb{C}$ such that $\varphi(f) \geq 0$ if $f \geq 0$ and $\|\varphi\| = 1$.*

The definition given above can be further appreciated if we also recall the well-known *Riesz-Markov* theorem:

Theorem 0.0.1. *Let X be a locally compact Hausdorff space and let φ be a positive linear functional on $C(X, \mathbb{R})$. Then there exists a unique Radon*[1] *(positive) measure μ on X such that*

$$\varphi(f) = \int_X f d\mu.$$

As is known, the regularity assumption on the measure is automatic for metrisable spaces. It is worth recalling that when X is compact any positive functional is automatically bounded and its norm is attained at 1, that is $\|\varphi\| = \varphi(\mathbb{1}) = \mu(X)$.

Now the physical interpretation attached to the real number $\varphi(f)$ is that it represents the expected value of the observable f in the state φ. More concretely, if one measures f many times and the system is in the state φ, then the average of this measures should be close to $\varphi(f)$. Bearing this intuition in mind, we can also define the variance of an observable with respect to a state:

[1]A Radon measure μ on a locally compact space X is a Borel measure such that:

1. μ is finite on compact subsets, i.e. $\mu(K) < \infty$ if $K \subset X$ is compact;
2. μ is inner regular, i.e. for any Borel set B one has $\mu(B) = \sup\{\mu(K) : K \text{ is compact and } K \subset B\}$;
3. μ is outer regular, i.e. for any Borel set B one has $\mu(B) = \inf\{\mu(O) : K \text{ is open and } O \supset B\}$.

Definition 0.0.5. *Let φ be a state and let $f \in \mathcal{O}$. Then, the variance of f with respect to φ is defined as*

$$\sigma_\varphi(f)^2 := \varphi\left((f-\varphi(f))^2\right) = \varphi(f^2) - \varphi(f)^2$$

It is easily checked that the variance is zero on all observables if and only if the state is pure, that is when it is a Dirac measure δ_x, for some $x \in X$.
The physical interpretation is that Dirac measures correspond to having no uncertainty whatsoever on our measurement, whereas general probability measures describe the more realistic situation in which the microscopic configuration of the system is only known partially.
Going back to the dynamics of our system, we saw above that the evolution of the states is described by the map φ. Now the interesting point is this dynamics can be transferred to observables as well since we can define a linear operator T_φ as

$$f \longmapsto T_\varphi f := f \circ \varphi$$

This is often referred to as the *Koopman operator* associated with φ. Unlike φ, the Koopman operator T_φ is always a linear map, which makes it possible to employ all typical tools coming from spectral theory.
When it comes to describing the statistical properties of a dynamical system, it is useful to consider the notion of *invariant measure*, namely a measure μ such that $\int_X f \mathrm{d}\mu = \int_X f \circ \varphi_t \, d\mu$ for any t, which is interpreted in the formalism we are outlining as an equilibrium state of our system.
This naturally leads to the well-known *ergodic hypothesis*, which was first formulated by Boltzmann. This is an assumption that in physics is often taken for granted.

For each initial state $x_0 \in X$ and each observable $f : X \to \mathbb{R}$ it is true that "time mean equals space mean", namely

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\varphi^n(x_0)) = \int_X f \, d\mu. \tag{1}$$

Among other things, the assumption (1) allows one to overcome the problem of knowing exactly what the initial state $x_0 \in X$ is. In fact, ergodic theory is actually concerned with studying invariant measures and their relation with large time behavior of dynamical systems. Therefore, the definition we gave above could be further refined in the following way.

Definition 0.0.6. *A classical dynamical system is a triple (X, φ_t, μ) where X is a (measurable) space of states, φ_t a dynamics on X, and μ an invariant measure for φ_t.*

The ergodic hypothesis will in general fail to hold true. Even so, it can be turned into mathematically rigorous theorems as soon as suitable hypotheses are made. A couple of classical results in this direction, due to Birkhoff and von Neumann, are worth mentioning (See [21] for more details).

Theorem 0.0.2 (Birkhoff's ergodic theorem). *Let (X, φ, μ) be a (discrete) classical dynamical system. Then, for any measurable $f : X \to \mathbb{R} \cup \{\infty\}$ that is finite μ-almost* *everywhere, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f \circ \varphi^n = g\,, \tag{2}$$

exists almost everywhere, where g is a $\varphi-$invariant function.

When the system is *ergodic*, namely if the only measurable φ-invariant functions are constant functions, then the ergodic hypotesis (1) follows at once as a direct application of the above theorem. At the Hilbert space level the ergodic theorem can be formulated as follows instead.

Theorem 0.0.3 (von Neumann's ergodic theorem). *Let U be a unitary operator on a Hilbert space $\mathcal{H}$ and E the orthogonal projection onto $\{\psi : U\psi = \psi\} = ker(I - U)$. Then we have*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n = E_{Ker(\mathbb{1}-U)} \tag{3}$$

where the limit is with respect to the strong operator topology. [2]

Notice that von Neumann's theorem applies in particular to the Koopman operator $U : L^2(X,\mu) \to L^2(X,\mu)$ associated with φ, i.e. $Uf := f \circ \varphi$ almost everywhere. However, its reach does not go beyond Birkhoff's ergodic theorem not least because for any fixed $f \in L^2(X,\mu)$ the convergence of the sequence of the means $\frac{1}{N+1}\sum_{n=0}^{N} f \circ \varphi^n$ is in the L^2-norm.

Now there is no reason to limit oneself to considering commutative C^*-algebras only, like $C(X)$. Quite the opposite, non-commutative C^*-algebras turn out to be the right tool to deal with quantum systems.

Definition 0.0.7. *The set of all observables $\mathcal{O}$ of a quantum system is given by the self-adjoint elements of a non-commutative C^*-algebra $\mathfrak{A}$.*

In a similar fashion, one can extend the concept of states to the quantum case:

Definition 0.0.8. *The set of states X of a quantum system is the set of all positive linear functionals ψ on $\mathfrak{A}$ such that $\|\psi\| = 1$.*

Example 0.0.3. Canonical Quantization
We try to give a quick and yet as comprehensive as possible description of a mathematical procedure known as first or canonical quantization, by means of which quantum systems with finetely many degree of freedoms can be settled. For simplicity, we limit ourselves to the case of a single (spinless) particle bounded on a line. We are thus considering a system with only one degree of freedom.
We denote by Q and P the observables corresponding to the position and the momentum,

[2] This theorem can be stated in more generality. Indeed, the result continues to hold for contractions, namely for bounded linear operators S with $\|S\| \leq 1$. The extension of the theorem to linear contractions can be obtained by applying Nagy's dilation theorem, which we discuss at lenght in Chapter 1.

respectively, of our particle. The starting point is to require the following commutation rule

$$QP - PQ := [Q, P] = iI$$

which is referred to as the *canonical commutation rule*. Unluckely, the canonical commutation rule cannot be derived either mathematically or physically but must be assumed as a new physical principle instead. Even so, a rather suggestive way to conceive it is to think of it as the quantum counterpart of the classical rule that $\{q, p\} = 1$, if $\{\cdot, \cdot\}$ is the Poisson bracket. More precisely, the so-called Dirac correspondence principle prescribes that the quantization of a classical system ought to be obtained by replacing Poisson brackets $\{\cdot, \cdot\}$ with $\frac{1}{i}[\cdot, \cdot]$.

As is known, the above commutation rule cannot be represented by bounded operators on a Hilbert space. However, a standard representation through unbounded operators does exist. This is the so-called Schroedinger representation: the Hilbert space is $L^2(\mathbb{R}, \mu_{\text{Leb}})$, while Q and P are given by

$$\begin{aligned}(Qf)(x) &= xf(x)\\(Pf)(x) &= -if'(x)\end{aligned}$$

defined on their natural domains of self-adjointness. It is then easily seen that $[Q, P] \subset iI$, that is the commutator between Q and P is extended by iI.

Now rather than dealing with the technicalities of unbounded operators, it is far preferable to consider the one-parameter groups of unitaries associated with Q and P. In other words, we consider $U(t) := e^{itQ}$ and $V(s) := e^{isP}$, for any $t, s \in \mathbb{R}$. In the Schroedinger representation, the two one-parameter groups are seen at once to be

$$\begin{aligned}U(t)f(x) &= e^{itx}f(x)\\V(s)f(x) &= f(x+s)\end{aligned}$$

In particular, they satisfy the following commutation rule

$$U(t)V(s) = e^{-its}V(s)U(t) \quad \text{for any } t, s \in \mathbb{R}$$

known as *Weyl relation*. This can be dealt with much more easily. Notably, the celebrated Stone-von Neumann theorem asserts that the Schroedinger representation is, up to unitary equivalence, the only irreducible representation of the Weyl relations. Furthermore, any representation decomposes into a direct sum of Schroedinger representations. By introducing a suitable locally compact group H, known as the Heisenberg group, any representation of the Weyl relation can be seen as a (strongly continuous) unitary representation of H. More importanly, the underlying C^*-algebra, whose self-adjoint elements represent the (bounded) observables of the system, turns out to be the algebra $K(\mathcal{H})$ of all compact operators acting on the separable Hilbert space $\mathcal{H}$, which in this formalism is identified with $C^*(H)$, the maximal group C^*-algebra of H.

Now if generalized observables are also taken into account by considering strong limits of genuine observable, the whole $B(\mathcal{H})$ is finaly arrived at. The states of this sytem are then identified with trace states: for any positive trace operator $T \in B(\mathcal{H})$ with $tr(T) = 1$, we consider $\varphi_T(A) := tr(AT)$. In particular, when T is a rank-one orthogonal projection, say $Tu := (u, x)x$, for some $x \in \mathcal{H}$ with $\|x\| = 1$, the corresponding state is nothing but

the vector state φ_x associated with x, that is $\varphi_x(A) := (Ax, x)$. Moreover, the set of vector states $\{\varphi_x : x \in \mathcal{H}_1\}$ is exactly the set of the extreme points of the convex set of all states. They represent those states on which the information on the system is as much as possible, which is why they are referred to as *pure states*.
The picture thus emerging is what in physics goes under the name of *Dirac formalism*. Rather interestingly, in the general framework provided by C^*-algebras this formalism can be fully recovered as a particular case when our C^*-algebra $\mathfrak{A}$ is required to have only one irreducible representation, which is the mathematical way to encode the physical request that superposition of pure states is always possible and yields a state that is still pure. In fact, a well-known theorem due to Rosenberg characterizes $K(\mathcal{H})$ as the sole C^*-algebra which only has one irreducible representation on a separable Hilbert space $\mathcal{H}$ up to unitary equivalence.

We can now move on to say what a C^*-dynamical sistem is in its full generality.

Definition 0.0.9. *A C^*-dynamical system is a triple $(\mathfrak{A}, G, \alpha)$ where $\mathfrak{A}$ is a C^*-algebra, G a locally compact group, and $\alpha : G \to Aut(\mathfrak{A})$ a group homomorphism, which is assumed to be strongly continuous, i.e. for any $a \in \mathfrak{A}$ the map $g \longmapsto \alpha(g)(a) \in \mathfrak{A}$ is norm continuous.*

It is worth noting that in physical applications the group G is nothing but the real line $\mathbb{R}$, which means the general definition can be replaced by the following:

Definition 0.0.10. *A C^*-dynamical system is a pair $(\mathfrak{A}, \Phi^t)$ $(t \in \mathbb{R})$ where $\mathfrak{A}$ is a unital C^*-algebra and Φ^t a one-parameter strongly continuous group of $*-$ automorphisms of $\mathfrak{A}$.*

That said, we would like to describe what we do in the present work. The thesis aims to analyze some spectral properties of the Koopman operator in the framework of commutative C^*-algebras, motivated by the study of the dynamics of a *dissipative* C^*-system.
A few words are now in order to explain what we mean by dissipative. Mathematically the definition is simply cast: the *-endomorphism Φ prescribing our dynamics is not invertible. The physical interpretation is that the dynamics fails to be reversible. In other terms, we can just follow the future evolution of the system without going back to its past.
Here follows a full description of a motivating example: the driven damped harmonic oscillator.

Example 0.0.4. It is well known that in presence of an external driving force, the *damped harmonic oscillator equation* becomes

$$m\ddot{x} + b\dot{x} + kx = F_0 cos(\omega t). \tag{4}$$

If we assume that $4km - b^2 > 0$, or in other terms if the friction is not prevailing over the elastic force, then the general solution of the homogeneous equation is

$$x(t) = Ce^{-\frac{b}{2m}t} cos\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t + \varphi\right) \tag{5}$$

which describes damped oscillations at frequency $\omega' := \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ within a decay envelope of time-dependent amplitude $e^{-\frac{b}{2m}t}$.

As for the complete equation, a straightforward computation shows that $\tilde{x}(t) = Acos(\omega t + \varphi_0)$ is a particular solution, where the coefficient A is given by

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

As is known, the greatest value of A is obtained when $\omega = \omega_0 := \sqrt{k/m}$, which corresponds to the so-called *resonance*.
To sum up, the general solution of the complete equation decomposes into a sum of two summunds $y(t) = x(t) + \tilde{x}(t)$, where $\lim_{t \to +\infty} x(t) = 0$. For this reason, $x(t)$ represents the so-called *transient* part of the solution, and it is obviously the part of the solution where the dynamics acts *dissipatevely*, whereas $\tilde{x}(t)$ is its *persistent* part.

Going back to the thesis, we start by reviewing in some detail the main properties of *completely positive maps*, which represent a far more general notion of morphisms between C^*-algebras. Even more interestingly, they can be used to mathematically frame several different physical situations, as is done with quantum channels from Quantum Information, to take but one example.
As we will see, thanks to the celebrated *Stinespring's dilation theorem*, it will always be possible to find a dilation of a completely positive map to a $*-$ homomorphism, or in terms of dynamical systems, to imbed a state-preserving C^*-dynamical system into a larger one under suitable conditions, as stated in [13].
That said, the problem we started from is to understand whether for a general dynamical system, whose dynamics is assigned through a completely potive map rather than a mere endomorphism, it is always possible to separate its persistent part from the dissipative part, in a similar fashion to what one does with the driven harmonic oscillator we recalled above.
Furthermore, it would also be interesting to see if the persistent part carries an intrinsic C^*-algebra structure, as one would expect. Lastly, it would be worth understanding if the restriction of the completely positive map Φ to its persistent part actually acts as a $^*-$automorphism. In the literature, this problem is known as *quantum decoherence* and is still an open issue of great interest treated by several authors such as [4], [16].
At any rate, we will limit ourselves to providing an example of interest and to studying some related spectral properties rather than facing the problem in its full generality. More precisely:

- In **Chapter 1** we provide a reasonably in-depth account of the theory of completely positive maps, which come in useful in dealing with more general physical situations, where endomorphisms are no longer enough.

- In **Chapter 2** we analyse the spectrum of the Koopman operator in the case of a commutative C^*-algebra $C(X)$. This is done by thoroughly reviewing E. Scheffold's work in [18].

- In **Chapter 3** we study the properties of a C^*-dynamical system, motivated by some considerations that come out from the previous chapter.

Admittedly, the primary aim of the thesis would have been to obtain the desired result working under more general hypotheses, as is done for instance in [4] and [16], where the map Φ is only supposed to be completely positive and no other assumptions are made. Despite our best efforts, though, achieving the result (3.3) when the map Φ is only assumed to be CP has revealed much harder than we actually expected and might well be a research problem to tackle afterwards.

Contents

CHAPTER1

Generalities on completely positive maps

In this chapter, after recalling some cornestones of C^*- algebra theory, we move on to give the notion of completely positive maps along with the main results and applications of interest. Most of the proofs of the results in the introductory part can be found in any introductory book dealing with the theory of C^*-algebras, such as [1], [15].

1.1 Basics of C^*-algebras

We start our discussion by recalling definitions and main properties of C^*-algebras.

Definition 1.1.1. *A complex normed algebra $\mathfrak{B}$ is an algebra over the complex field $\mathbb{C}$ endowed with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\| \quad \forall a, b \in \mathfrak{B}$. Moreover, if $\mathfrak{B}$ has a unit $\mathbb{1}$, we ask that $\|\mathbb{1}\| = 1$.*

Definition 1.1.2. *A Banach algebra $\mathfrak{B}$ is a normed algebra that is complete with respect to the metric induced by its norm.*

Definition 1.1.3. *An involution on a complex algebra $\mathfrak{B}$ is an anti-linear map $^* : \mathfrak{B} \to \mathfrak{B}$ such that:*

- ** is anti-multiplicative, that is $(ab)^* = b^*a^*$ for every $a, b \in \mathfrak{B}$*
- ** is involutive, that is $b^{**} = b$ for every $a \in \mathfrak{B}$*

A complex algebra that also carries such an involution is often referred to as an involutive algebra or a *-algebra.

Definition 1.1.4. *A Banach *-algebra $\mathfrak{B}$ is an involutive Banach algebra such that the involution is isometric, namely $\|b^*\| = \|b\|$ for every $b \in \mathfrak{B}$.*

We are now ready to say what a C^*-algebra is.

Definition 1.1.5. *A C^*-algebra $\mathfrak{A}$ is an involutive Banach algebra whose norm $\|\cdot\|$ satisfies the following equality $\|a^*a\| = \|a\|^2$, $\forall a \in \mathfrak{A}$, known as the C^* identity.*

There follows a couple of remarks.

Remark 1.1.1. The C^*-identity actually implies that the involution is automatically isometric. Indeed, we have $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, hence $\|a\| \leq \|a^*\|$. But then we also have $\|a^*\| \leq \|a^{**}\| = \|a\|$, which means $\|a^*\| = \|a\|$.

Remark 1.1.2. Given a C^*-algebra $\mathfrak{A}$, it may not have an identity $\mathbb{1}$. However, it is always possible to add it, by defining $\mathfrak{A}_1 = \mathfrak{A} \oplus \mathbb{C}\mathbb{1}$, equipped with norm

$$\|a \oplus \lambda\mathbb{1}\| = \sup_{b\in\mathfrak{A}\,:\,\|b\|=1} \|ab + \lambda b\|$$

For brevity we omit the proof that the norm thus defined satisfies the C^*-identity.

Thanks to the latter remark, in the following we may as well consider unital algebras only, unless otherwise specified.

Definition 1.1.6. *Let $\mathfrak{A}$ be a C^*-algebra and $a \in \mathfrak{A}$. The spectrum of a is defined as*

$$\sigma_{\mathfrak{A}}(a) = \{\lambda \in \mathbb{C} \,:\, \nexists (a - \lambda\mathbb{1})^{-1} \ (in\ \ \mathfrak{A})\}.$$

Remark 1.1.3. The spectrum of an element $a \in \mathfrak{A}$ is a non-empty compact subset of $\mathbb{C}$.

Remark 1.1.4. The spectral radius of an element $a \in \mathfrak{A}$ is defined as $r(a) := sup\{|\lambda| : \lambda \in \sigma_{\mathfrak{A}}(a)\}$

Remark 1.1.5. The *resolvent* set of $a \in \mathfrak{A}$ is defined as $\rho_{\mathfrak{A}}(a) := \mathbb{C} \setminus \sigma_{\mathfrak{A}}(a)$ and the resolvent operator of a is the (analytic) map

$$\begin{aligned} R_a \,:\, \rho_{\mathfrak{A}}(a) &\to \mathfrak{A} \\ \lambda &\mapsto (\lambda\mathbb{1} - a)^{-1} \end{aligned}$$

Remark 1.1.6. In particular, we recall that if $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ a C^*-subalgebra of $\mathfrak{A}$, then $\sigma_{\mathfrak{A}}(a) = \sigma_{\mathfrak{B}}(a)$ for $a \in \mathfrak{A}$, which means the subscript can be safely left out.

Remark 1.1.7. If $\mathfrak{B}$ is a Banach algebra, $a \in \mathfrak{B}$, $s > r(A)$ and $\mathbb{D}_s = \{\lambda \in \mathbb{C} \,:\, |\lambda| < s\}$, then the map

$$\Phi \,:\, \mathrm{H}(\mathbb{D}_s) \ni f \longmapsto \frac{1}{2\pi i}\int_{\gamma_r} f(z)R(z,a)dz$$

where $0 < r < s$ is such that still $\sigma(A) \subseteq \mathbb{D}_r$ and γ_r is the contour which traverses the boundary of $\mathbb{D}_r$ once in counterclockwise direction, is called *Dunford-Riesz Calculus.*

Remark 1.1.8. If $\mathfrak{A}$ is an commutative C^*-algebra, its spectrum, denoted with $\sigma(\mathfrak{A})$, is a subset of $\mathfrak{A}^*$, precisely contained in the dual unit ball $\mathfrak{A}^*_1$, and it is compact under the weak*- topology.

In particular, we recall an important result which states that any commutative C^*-algebra can be identified with an algebra of complex-valued functions on a compact Hausdorff space X, that is the spectrum of the algebra.

Theorem 1.1.1 (Gelfand-Naimark). *If $\mathfrak{A}$ is a commutative C^*-algebra, then it is $^*-$isometrically isomorphic to $C(X)$, where X is a compact Hausdorff space.*

Definition 1.1.7. *A representation of a C^*-algebra is a $*-$ homomorphism $\pi : \mathfrak{A} \to B(\mathcal{H})$. Two representatons $\pi_1 : \mathfrak{A} \to B(\mathcal{H}_1)$, $\pi_2 : \mathfrak{A} \to B(\mathcal{H}_2)$ are (unitarily) equivalent, if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$U\pi_1(a) = \pi_2(a)U, \quad a \in \mathfrak{A}.$$

Remark 1.1.9. Denote with $\mathfrak{A}^+ = \{a^*a \; : \; a \in \mathfrak{A}\}$ the cone of positive elements of $\mathfrak{A}$. Equivalently, $a \in \mathfrak{A}^+$ if and only if it is self-adjoint ($a = a^*$) and $\sigma(a) \subseteq [0, \infty)$.

Definition 1.1.8. *A linear functional on a C^*-algebra $f : \mathfrak{A} \to \mathbb{C}$ is positive, if $f(a^*a) \geq 0$, i.e., if $f(a^*a) \subseteq \mathbb{R}_{\geq 0}$.*

Remark 1.1.10. For a positive linear functional φ on a C^*-algebra, it holds $\|\varphi\| = \varphi(\mathbb{1})$.

Remark 1.1.11. The positive linear functionals of norm one are known as the *states* of $\mathfrak{A}$, denoted by $S_{\mathfrak{A}}$. These form a compact convex subset of $\mathfrak{A}^*$ in the weak*- topology.

The above notions are fundamental to state the famous GNS construction:

Theorem 1.1.2. *Given a positive functional $\varphi : \mathfrak{A} \to \mathbb{C}$ there is a cyclic representation $\pi_\varphi : \mathfrak{A} \to B(\mathcal{H}_\varphi)$ with generator $\xi_\varphi \in \mathcal{H}_\varphi$ such that*

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle. \tag{1.1}$$

The basic idea is to use a positive functional $\varphi : \mathfrak{A} \to \mathbb{C}$ to turn a quotient of the left regular representation of a C^*-algebra into a representation.

Remark 1.1.12. For every self-adjoint $a \in \mathfrak{A}$, there is a state φ such that $|\varphi(a)| = \|a\|$.

Theorem 1.1.3 (Gelfand-Naimark II). *Any C^*-algebra is $^*-$ isometrically isomorphic with a norm-closed $*-$ subalgebra of $B(\mathcal{H})$.*

1.2 Positive and completely positive maps

Some problems arising (as in Quantum Information for instance) from quantum physics show that in general endomorphims of C^*-algebras may not be enough to cover all possible situations one would like to deal with. One way to overcome this limit is to introduce a more general class of maps, known as completely positive maps. In the following we provide a brief but sufficiently detailed account of their main properties. To provide a self-contained treatment, though, we first need to recall some very basic definitions. We start by characterizing the space $M_n(\mathfrak{A})$, of $n \times n$ matrices with entries from a C^*-algebra $\mathfrak{A}$ (one could check [14] for more details). It's well established that the above space, with the usual operations of matrix multiplication and transpose, namely,

$$(a_{ij}) \cdot (b_{ij}) = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)$$
$$(a_{ij})^* = (a_{ji}^*)$$

is a $^*-$ algebra, with (a_{ij}), $(b_{ij}) \in M_n(\mathfrak{A})$. Furthermore, one can proceed by considering the most basic of C^*-algebras $B(\mathcal{H})$ and providing an identification for $M_n(B(\mathcal{H}))$. This is accomplished by considering the map

$$\begin{gathered}(T_{ij}) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)} \\ (T_{ij}) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} := \begin{bmatrix} \sum_{j=1}^n T_{1j} h_j \\ \vdots \\ \sum_{j=1}^n T_{nj} h_j \end{bmatrix}\end{gathered}$$

for an element $(T_{ij}) \in M_n(B(\mathcal{H}))$, with $\mathcal{H}^{(n)}$ denoting n copies of the hilbert space $\mathcal{H}$

$$\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}.$$

At this point, it's trivial to verify that the above operator is bounded in $\mathcal{H}^{(n)}$ and that there exists a $* -$ *isomorphism* between $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^{(n)})$ which provides us the desired identification. For a general C^*-algebra $\mathfrak{A}$, it will be sufficient to consider a $^*-$ representation $\pi : \mathfrak{A} \to B(\mathcal{H})$, and consequently, $\pi : M_n(\mathfrak{A}) \to M_n(B(\mathcal{H})$ whose image will be closed by completeness of $M_n(\mathfrak{A})$, and hence a C^*-algebra. Now we're ready to give the definition, of *positive* and *completely positive* maps, respectively:

Definition 1.2.1. *Given $\mathfrak{A}$, $\mathfrak{B}$ two C^*-algebras, a map $\phi : \mathfrak{A} \to \mathfrak{B}$ is positive, if*

$$\forall a \in \mathfrak{A}^+ \Rightarrow \phi(a) \in \mathfrak{B}^+ \tag{1.2}$$

where $\mathfrak{A}^+$, $\mathfrak{B}^+$ denote the closed convex cones of positive elements of $\mathfrak{A}$ and $\mathfrak{B}$.
The map ϕ is called completely positive, *if*

$$\begin{gathered}\phi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B}) \\ \phi_n((a_{ij})) := (\phi(a_{ij}))\end{gathered} \tag{1.3}$$

is positive for each $n \in \mathbb{N}$.

One could initially hope that C^*-algebras are sufficiently "nice" that any positive map is also completely positive, but easy examples are available to check that this does not hold, in general: let $\{E_{i,j}\}_{i,j=1}^n$ the system of matrix units for M_2 ($E_{i,j} \equiv 1$ in the $(i,j)-th$ entry and zero elsewhere), and let $\phi : M_2 \to M_2$ the transpose map, namely $\phi(E_{i,j}) = E_{j,i}$. It's easy to prove that the transpose of a positive matrix is positive. Now let's consider $\phi_2 : M_2(M_2) \to M_2(M_2)$.
Note that, the matrix of matrix units

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is positive, but

$$\phi_2 \left[\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \right] = \begin{bmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive. Therefore, ϕ is a positive map but not completely positive.
We will provide a deeper characterization of completely positive maps in the *Stinespring theorem.*

1.3 Dilations

1.3.1 Unitary dilation of an isometry

Now, we briefly introduce the topic of *dilations*: the main idea consists of realizing a mapping into an operator space as "part" of something simpler on a larger space. The first case concerns the *unitary dilation of isometry*: let $V : \mathcal{H} \to \mathcal{H}$ be an isometry and $P := I_{\mathcal{H}} - VV^*$ be the projection onto the orthocomplement of the range of V, if we define U on $\mathcal{H} \oplus \mathcal{H}$ via

$$U = \begin{bmatrix} V & P \\ 0 & V^* \end{bmatrix},$$

then U is unitary since $U^*U = UU^* = I_{\mathcal{K}}$. Moreover, by the identification

$$\begin{aligned} \mathcal{H} &\longleftrightarrow \mathcal{H} \oplus 0 \\ u &\longleftrightarrow (u, 0) \end{aligned}$$

then

$$V^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \forall n \geq 0. \tag{1.4}$$

So, any isometry V can be realized as the restriction of some unitary to one of its subspaces in a manner that also respects the powers of both operators.

1.3.2 Isometric dilation of a contraction

In a similar way, one could also construct an *isometric dilation of contraction.*
Let T be a contraction operator on $\mathcal{H}$, and let $D_T = (I - T^*T)^{1/2}$. We set

$$\ell^2(\mathcal{H}) = \left\{ (h_1,\, h_2, \dots) : h_n \in \mathcal{H} \ \forall n\,,\, \sum_{n=1}^{\infty} \|h_n\|^2 < +\infty \right\}.$$

This is a Hilbert space with norm $\|(h_1, h_2, \dots)\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$ and inner product $\langle (h_1, h_2, \dots), (k_1, k_2, \dots) \rangle = \sum_{n=1}^{\infty} \langle h_n, k_n \rangle$. As before, we define now

$$\begin{aligned} V : \ell^2(\mathcal{H}) &\longrightarrow \ell^2(\mathcal{H}) \\ V(h_1, h_2 \dots) &:= (Th_1, D_T h_1, h_2, \dots), \end{aligned}$$

which is an isometry on $\ell^2(\mathcal{H})$ since

$$\|Th\|^2 + \|D_T h\|^2 = \langle T^*Th, h \rangle + \langle D_T^2 h, h \rangle = \|h\|^2.$$

Identifying now $\mathcal{H}$ with $\mathcal{H} \oplus 0 \oplus \dots$, it's clear that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}. \tag{1.5}$$

At this point, one can put these two constructions together, in order to obtain the *unitary dilation of contraction*, that is

Theorem 1.3.1 (**Sz.-Nagy's dilation theorem**). *Let T a contraction operator on a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary operator U on $\mathcal{K}$ such that*

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \tag{1.6}$$

Proof. We define $\mathcal{K}$ as $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$ and identify $\mathcal{H}$ with $(\mathcal{H} \oplus 0 \oplus \dots) \oplus 0$. Let V the isometric dilation of T and U be the unitary dilation of V on $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$. Since $\mathcal{H} \subseteq \ell^2(\mathcal{H}) \oplus 0$, we have that

$$P_{\mathcal{H}} U^n|_{\mathcal{H}} = P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n \quad \forall n \geq 0.$$

□

To see the power of this simple geometric construction, we now provide Sz.-Nagy's proof of the *von Neumann inequality.*

Proposition 1.1 (**von Neumann's inequality**). *Let T be a contraction operator on a Hilbert space $\mathcal{H}$. Then for any polynomial p,*

$$\|p(T)\| \leq \sup\{|p(z)| \, : \, |z| \leq 1\} \tag{1.7}$$

Proof. Let U be the unitary dilation of T given by by (1.3.1).
Since $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ $\forall n \geq 0$, by taking linear combinations, that $p(T) = P_{\mathcal{H}} \, p(U)|_{\mathcal{H}}$ and hence $\|p(T)\| \leq \|p(U)\|$. Since unitaries are normal operators, we have that $\|p(U)\| = sup\{|p(\lambda)| \, : \, \lambda \in \sigma(U)\}$. The result immediately follows, since U is unitary and its spectrum is contained in the unit disk. □

Now we finally give the following result, which is a structure theorem for completely positive maps. The proof is taken again from [14], and more details have been added.

Theorem 1.3.2 (**Stinespring**). *Let $\mathfrak{A}$ a unital C^*- Algebra and $\Phi : \mathfrak{A} \to B(\mathcal{H})$ a completely positive map. Then there exists $\mathcal{K} \supset \mathcal{H}$ as subspace, a unital *-homomorphism $\pi : \mathfrak{A} \to B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ such that*

$$\|V\|^2 = \|\Phi(\mathbb{1})\| \quad , \quad \Phi(a) = V^* \pi(a) V. \tag{1.8}$$

Proof. Consider the algebraic tensor product $\mathfrak{A} \otimes \mathcal{H} = \{a \otimes h \, : \, a \in \mathfrak{A} \, , \, h \in \mathcal{H}\}$ and define a bilinear form on this space

$$\langle a \otimes x, b \otimes y \rangle := \langle \Phi(b^* a) x, y \rangle_{\mathcal{H}}$$

Now, the fact that Φ is completely positive ensures that the above bilinear form is positive semidefinite, since by linear extension,

$$\sum_{i,j=1}^{n} \langle a_j \otimes x_j, a_i \otimes x_i \rangle = \sum_{i,j=1}^{n} \langle \Phi(a_i^* a_j) x_j, x_i \rangle = \left\langle \Phi_n((a_i^* a_j)) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\rangle_{\mathcal{H}^{(n)}} \geq 0.$$

Positive semidefinite bilinear forms satisfy the *Cauchy-Schwartz inequality*

$$|\langle u,v\rangle|^2 \leq \langle u,u\rangle \cdot \langle v,v\rangle.$$

Thus we can define a subspace of $\mathfrak{A}\otimes\mathcal{H}$ as

$$\mathcal{N} := \{u \in \mathfrak{A}\otimes\mathcal{H} \mid \langle u,u\rangle = 0\} = \{u \in \mathfrak{A}\otimes\mathcal{H} \mid \langle u,v\rangle = 0 \ \ \forall v \in \mathfrak{A}\otimes\mathcal{H}\},$$

and consider the induced bilinear form on quotient space $\mathfrak{A}\otimes\mathcal{H}/\mathcal{N}$ defined in the usual way

$$\langle u+\mathcal{N}, v+\mathcal{N}\rangle := \langle u,v\rangle$$

that will be an inner product, and we let denote $\mathcal{K}$ the Hilbert space that is the completion of the inner product space $\mathfrak{A}\otimes\mathcal{H}/\mathcal{N}$. If $a\in\mathfrak{A}$ we also define a linear map $\pi(a) : \mathfrak{A}\otimes H \to \mathfrak{A}\otimes\mathcal{H}$ by

$$\pi(a)\left(\sum_{i=1}^{n} a_i\otimes x_i\right) := \sum_{i=1}^{n}(aa_i)\otimes x_i,$$

namely, the *left a-multiplication.* Consequently,

$$\begin{aligned}
&\left\langle \pi(a)\left(\sum_{j=1}^{n} a_j\otimes x_j\right), \pi(a)\left(\sum_{i=1}^{n} a_i\otimes x_i\right)\right\rangle = \\
&= \sum_{i,j=1}^{n}\langle\Phi(a_i^* a^* a a_j)x_j, x_i\rangle_{\mathcal{H}} \leq \|a^*a\|\cdot\sum_{i,j=1}^{n}\langle\Phi(a_i^* a_j)x_j, x_i\rangle_{\mathcal{H}} \\
&= \|a\|\cdot\left\langle \sum_{j=1}^{n} a_j\otimes x_j, \sum_{i=1}^{n} a_i\otimes x_i\right\rangle,
\end{aligned}$$

where we have used the inequality $(a_i^* a^* a a_j) \leq \|a^*a\|\cdot(a_i^* a_j)$.
Therefore, we derive some information:

- $\pi(a)$ leaves $\mathcal{N}$ invariant, and therefore induces a quotient linear transformaton on $\mathfrak{A}\otimes\mathcal{H}/\mathcal{N}$ which we still denote by $\pi(a)$.
- $\|\pi(a)\| \leq \|a\|$, hence $\pi(a)$ extends to a bounded linear operator on $\mathcal{K}$.

At this point, it's pretty easy to show that

$$\begin{aligned}
\pi : \mathfrak{A} &\longrightarrow B(\mathcal{K}) \\
a &\longmapsto \pi(a)(c\otimes x+\mathcal{N})
\end{aligned}$$

is a unital $*$-homomorphism, since

$$abc\otimes x+\mathcal{N} = \pi(ab)(c\otimes x+\mathcal{N}) = \pi(a)(bc\otimes x+\mathcal{N}) = \pi(a)\pi(b)(c\otimes x+\mathcal{N})$$

and

$$\begin{aligned}
\langle\pi(a)^*(c\otimes x), s\otimes y\rangle_{\mathcal{K}} &= \langle c\otimes x, as\otimes y\rangle_{\mathcal{K}} \\
&= \langle a^*c\otimes x, s\otimes y\rangle_{\mathcal{K}} \\
&= \langle\Phi((as)^*c)x, y\rangle_{\mathcal{H}}
\end{aligned}$$

and the same holds for $\pi(a^*)$, thus $\pi(a^*) = \pi(a)^*$.
Finally, define $V : \mathcal{H} \to \mathcal{K}$ via

$$V(x) = \mathbb{1} \otimes x + \mathcal{N}.$$

Then V is bounded, because

$$\|Vx\|^2 = \langle \mathbb{1} \otimes x, \mathbb{1} \otimes x \rangle = \langle \Phi(\mathbb{1})x, x \rangle_{\mathcal{H}} \leq \|\Phi(\mathbb{1})\| \cdot \|x\|^2.$$

Indeed, it is clear that $\|V\|^2 = sup\{\langle \Phi(\mathbb{1})x, x\rangle_{\mathcal{H}} \mid \|x\| \leq 1\} = \|\Phi(\mathbb{1})\|$. To complete the proof, we only need to see that

$$\langle V^*\pi(a)Vx, y\rangle_{\mathcal{H}} = \langle \pi(a)\mathbb{1} \otimes x, \mathbb{1} \otimes y\rangle_{\mathcal{K}} = \langle \Phi(a)x, y\rangle_{\mathcal{H}}\,,$$

for all x, y, and so $V^*\pi(a)V = \Phi(a)$. □

There are several remarks we can make to this theorem. First, any map of the form $\Phi(a) = V^*\pi(a)V$ is easily seen to be completely positive. Thus, Stinespring's dilation fully characterizes the completely positive maps from any C^*-algebra into the algebra of bounded operators on a Hilbert space $\mathcal{H}$. Second, if Φ is unital, then V is an isometry. In this case we may identify $\mathcal{H}$ with the subspace $V\mathcal{H}$ of $\mathcal{K}$. By means of this identification, VV^* becomes the projection of $\mathcal{K}$ onto $\mathcal{H}$, which we denote by $P_{\mathcal{H}}$. Therefore, according to a language that is common in dilation theory, one can write

$$\Phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}.$$

meaning that the completely positive map Φ is nothing but a *compression* of a $*-$ homomorphism in $B(\mathcal{K})$.

1.4 C^*-system embeddings

At this point, once we have established that any completely positive map admits a natural dilation to a $*-$ homomorphism it is certainly interesting to analyze the following issue from the point of view C^*-dynamical systems: in particular, the authors of [13] investigated the possibility of imbedding a state preserving C^*-dynamical system into a larger one, and the answer is given by the so-called *GNS covariant representation.* In accordance with [8], we start remarking some preliminaries.
A (discrete) C^*-dynamical system is a pair $(\mathfrak{A}, \Phi)$ consisting of a C^*-algebra and a positive map $\Phi : \mathfrak{A} \to \mathfrak{A}$. Notice that, if $\|\Phi\| = 1$, which happens if Φ is completely positive and $\|\Phi(\mathbb{1})\| = 1$, then $\sigma(\Phi) \subset \overline{\mathbb{D}}$. The part of the spectrum $\sigma(\Phi) \bigcap \mathbb{T}$ living on the unit circle is called *peripheral.*

Proposition 1.2. *With a slight abuse of notation, one can denote by the triplet* $(\mathfrak{A}, \Phi, \varphi)$ *the* C^**-dynamical system at issue, pointing out that* φ *is an invariant state under the dynamics generated by* Φ*, and consider the associated GNS construction* $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$*. If, in addition,*

$$\varphi(\Phi(a)^*\Phi(a)) \leq \varphi(a^*a) \quad , a \in \mathfrak{A},$$

then there exists a unique contraction $U_{\varphi,\Phi} \in B(\mathcal{H}_\varphi)$ *such that* $U_{\varphi,\Phi}\,\xi_\varphi = \xi_\varphi$ *and*

$$U_{\varphi,\Phi}\,\pi_\varphi(a)\xi_\varphi = \pi_\varphi(\Phi(a))\xi_\varphi \quad , a \in \mathfrak{A}.$$

The quadruple $(\mathcal{H}_\varphi, \pi_\varphi, U_{\varphi,\Phi}, \xi_\varphi)$ is called the covariant GNS representation associated to $(\mathfrak{A}, \Phi, \varphi)$. Also, if Φ is a $^*-$homomorphism, then $U_{\varphi,\Phi}$ is an isometry with range-projection $U_{\varphi,\Phi}U^*_{\varphi,\Phi}$, the orthogonal projection onto the subspace $\overline{\pi_\varphi(\Phi(\mathfrak{A}))\xi_\varphi}$ (See [8] and Lemma 2.1, Propositions 6.1, 6.2 from [13]). At this point, we can state the following result, which will clarify the nature of embedding we have announced.

Theorem 1.4.1. *Let $(\mathfrak{A}, \varphi, \Phi)$ be a state preserving C^*-dynamical system and let $U_{\Phi,\varphi}$ denote the linear isometry defined on $\mathcal{H}_\varphi$ by $U_{\Phi,\varphi}(\pi_\varphi(x)\xi_\varphi) = \pi_\varphi(\Phi(x))\xi_\varphi$, $x \in \mathfrak{A}$. Let further $\tilde{U}_{\Phi,\varphi}$be the minimal unitary dilation of $U_{\Phi,\varphi}$. Then, if we define*

$$
\begin{aligned}
\tilde{\pi}_{\Phi,\varphi} : \mathfrak{A} \ni a \mapsto & \, V_{\Phi,\varphi}\pi_\varphi(a)V^*_{\Phi,\varphi} \\
& + \sum_{k=1}^{\infty} \tilde{U}^{-k}_{\Phi,\varphi} V_{\Phi,\varphi}(\pi_\varphi(\Phi^k(a))(\mathbb{1}_{\mathcal{H}_\varphi} - U_{\Phi,\varphi}U^*_{\Phi,\varphi})V^*_{\Phi,\varphi}\tilde{U}^k_{\Phi,\varphi} \\
& = so - \lim_{n\to\infty} \tilde{U}^{-n}_{\Phi,\varphi} V_{\Phi,\varphi}\pi_\varphi(\Phi^n(a))V^*_{\Phi,\varphi}\tilde{U}^n_{\Phi,\varphi}.
\end{aligned}
$$

with the convergence of the series being understood in the strong operator topology, where $V_{\Phi,\varphi} : \mathcal{H}_\varphi \to \tilde{\mathcal{H}_\varphi}$ is the isometry provided by Nagy's dilation theorem, we have that $\tilde{\pi}_{\Phi,\varphi} : \mathfrak{A} \to B(\tilde{\mathcal{H}_\varphi})$ is a $*$-representation and the following equality holds

$$
\tilde{\pi}_{\Phi,\varphi}(\Phi(a)) = \tilde{U}_{\Phi,\varphi}\tilde{\pi}_{\Phi,\varphi}(a)\tilde{U}^*_{\Phi,\varphi}. \tag{1.9}
$$

Chapter2

The spectrum of some CP maps

In this chapter, our attention is focussed on some spectral properties of particular cases of CP maps, namely endomorphisms of commutative C^*-algebras or finitely dimensional C^*-algebras.

We start by recalling a couple of interesting results, both due to Stinespring, which show that complete positivity automatically follows from positivity for commutative C^*-algebras.

Theorem 2.0.1. *Let $\mathfrak{B}$ a C^*-algebra and $\Phi : C(X) \to \mathfrak{B}$ a positive map. Then Φ is completely positive.*

Proof. For a detailed proof, the reader is referred to [14]. □

The result continues to hold when it is the range of Φ to be a commutative C^*-algebra. More precisely, we have:

Theorem 2.0.2. *Let $\mathcal{S}$ an operator system and $\Phi : \mathcal{S} \to C(X)$ a bounded linear map. If Φ is positive, then it is completely positive.*

Proof. Again, for a detailed proof the reader is referred to [14]. □

Remark 2.0.1. If $\mathcal{S}$ is a subset of a C^*-algebra $\mathfrak{A}$, then we set

$$\mathcal{S}^* = \{a \,:\, a^* \in \mathcal{S}\}$$

and we say $\mathcal{S}$ is self-adjoint when $\mathcal{S} = \mathcal{S}^*$. If $\mathfrak{A}$ has unit $\mathbb{1}$ and $\mathcal{S}$ is a self-adjoint subspace of $\mathfrak{A}$ containing $\mathbb{1}$, then $\mathcal{S}$ is said to be an *operator system.*
In the following discussion, therefore, the concepts of positivity and complete positivity come to coincide, allowing us to use effective tools, coming from the commutative framework, for the analysis of spectral properties.

2.1 Decomposition of the spectrum of a bounded operator

If X is a Banach space, $T \in B(X)$, it is often useful to write $\sigma(T)$ as a union of three particular disjoint subsets:

- The *point spectrum* $\sigma_p(T)$, also referred to as the *pure point spectrum*, is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda\mathbb{1}$ is not injective. Equivalently, $\lambda \in \sigma_p(T)$ if it is an *eigenvalue* of T. The non-trivial subspace $\mathrm{Ker}(T - \lambda\mathbb{1})$ is then known as the eigenspace associated with the eigenvalue λ.

- The *continuous spectrum* $\sigma_c(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T-\lambda\mathbb{1}$ is injective, has dense range, but nevertheless it fails to be surjective.

- The *residual spectrum* $\sigma_r(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda\mathbb{1}$ is injective but does not have dense range.

Remark 2.1.1. It is worth recalling that the residual spectrum of any self-adjoint operator on a Hilbert space is always empty.

It is apparent that the spectrum of T decomposes into the disjoint union of the three sets above, that is

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Actually, a more general notion of eigenvalue is also possible, as the next definition shows.

Definition 2.1.1 (Approximate point spectrum). *Let X be a Banach space and $T \in B(X)$. The set*

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists \{u_n\}_{n\in\mathbb{N}} \quad \text{such that} \quad \|Tu_n - \lambda u_n\| \to 0 \quad \text{as } n \to \infty\} \tag{2.1}$$

where $\|u_n\| = 1$ $\forall n \in \mathbb{N}$, is called the approximate point spectrum of T.

Remark 2.1.2. The definition of approximate spectrum is worthy of a comparison with that of *essential spectrum* for an operator T acting on a Hilbert space. This is defined as the set $\sigma_{ess}(T) \subset \sigma(T)$ of those values λ such that there exists an orthonormal infinite set $\{e_j : j \in \mathbb{N}\}$ such that $\lim_{j\to\infty} \|(T - \lambda\mathbb{1})e_j\| = 0$. Obviously $\sigma_{ess}(T)$ is contained in $\sigma_{ap}(T)$. More importantly, the essential spectrum can be shown to be invariant under compact perturbations. Indeed, it can also be characterized as the spectrum of T thought of as an element of the Calkin algebra, i.e. the spectrum of $\pi(T)$ in the quotient C^*-algebra $B(\mathcal{H})/K(\mathcal{H})$, where $\pi : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$ is the projection onto the quotient. As an additional property, it is important to remember that the essential spectrum is closed and non-empty.

In the next result we prove a couple of remarkable topological properties enjoyed by the approximate spectrum, which will come in useful in the sequel.

Lemma 2.1. *Let X be a Banach space and $T \in B(X)$, then*
i) The approximate point spectrum $\sigma_{ap}(T)$ is a closed non-empty subset of $\sigma(T)$.
ii) $\partial\sigma(T) \subset \sigma_{ap}(T)$.

Proof. i) If $\lambda \notin \sigma(T)$ then by the bounded inverse theorem $(T - \lambda I)$ admits a bounded inverse, and in particular

$$\|(T - \lambda I)x\| \geq c\|x\|$$

where $c := \|(T-\lambda I)^{-1}\|^{-1}$. This means that $(T-\lambda I)$ is bounded from below ($\lambda \notin \sigma_{ap}(T)$), and hence $\sigma_{ap}(T) \subset \sigma(T)$. To show that $\sigma_{ap}(T)$ is closed, fix $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$, and taking again the constant c as above we have $\|(T - \lambda I)x\| \geq c\|x\|$. Now fix $\lambda_1 \in \mathbb{C}$ such that $|\lambda_1 - \lambda| < c$, we have

$$\begin{aligned}\|Tx - \lambda_1 x\| = \|Tx - (\lambda_1 - \lambda)x - \lambda x\| &\geq \|Tx - \lambda x\| - |\lambda_1 - \lambda|\|x\| \\ &\geq (c - |\lambda_1 - \lambda|)\|x\|.\end{aligned}$$

Thus $\sigma_{ap}(T)$ has an open complement and hence is closed.
ii) Now, let $\lambda \in \partial\sigma(T)$ and let $\varepsilon > 0$. Since $\sigma(T)$ and $\mathbb{C} \setminus \sigma(T)$ have the same boundary, then there is $\mu \notin \sigma(T)$ such that $|\lambda - \mu| < \frac{\varepsilon}{2}$. We have

$$\frac{2}{\varepsilon} \leq \frac{1}{d(\mu)} \leq \|(T - \mu I)^{-1}\|$$

where $d(\mu)$ denotes the distance from μ to the spectrum of T, since it is a well known fact that

$$\|R_T(\lambda)\| \to \infty \quad as \quad d(\lambda) \to 0.$$

Therefore, there is a $x \in X$ such that $\|x\| = 1$ and

$$\frac{1}{\varepsilon} \leq \|(T - \mu I)^{-1}x\|.$$

Let $y = \frac{1}{\|(T-\mu I)^{-1}x\|}(T - \mu I)^{-1}x$, then $\|y\| = 1$ and

$$\|(T - \lambda I)y - (T - \mu I)y\| < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}\|(T - \lambda I)y\| &\leq \|(T - \lambda I)y - (T - \mu I)y\| + \|(T - \mu I)y\| \\ &< \frac{\varepsilon}{2} + \frac{\|x\|}{\|(T - \mu I)^{-1}\|} \\ &< \frac{3\varepsilon}{2}.\end{aligned}$$

and hence it results $\lambda \in \sigma_{ap}(T)$. Since a non-empty compact subset of $\mathbb{C}$ has non-empty boundary, then $\sigma_{ap}(T)$ is non-empty. □

2.2 The spectrum of an isometry

We now characterize the spectrum of an isometry on a Banach space X, based on the assumption that we have at least one point lying in the inner part of the spectrum. That is, we assume that the isometry is not surjective.

Remark 2.2.1. In the context of Hilbert spaces, we exactly know how isometries look like, thanks to the known *Wold's decomposition*, which states that every isometry V takes the form $V = (\oplus_{\alpha \in A} S) \oplus U$ for some index set A, where S is the unilateral shift on a separable Hilbert space $\mathcal{H}_\alpha$, and U is a unitary operator.

The final part of the proof of the following proposition was taken by [23], and we added the missing initial part.

Theorem 2.2.1. *Let X be a Banach space and $T \in B(X)$ an isometry. Then, either of the two conditions holds:*

- *if T is surjective, then $\sigma(T) \subset \mathbb{T}$.*
- *if T is not surjective, then $\sigma(T) = \overline{\mathbb{D}}$.*

Proof. The first case can be dealt with as in Remark 2.5.3. The second case requires more work to do. If we set $C := \sigma(T)$, then C is a closed set containing 0 by hupothesis and contained in $\overline{\mathbb{D}}$. We want to show that the inclusion $C \subset \overline{\mathbb{D}}$ cannot be proper. We shall argue by contradiction. Suppose C is a proper subset of the closed disk $\overline{\mathbb{D}}$, and let A be the complement of C, *i.e.* $A = \mathbb{C} \setminus C$.
Let us denote by A_∞ the complement of $\overline{\mathbb{D}}$, *i.e.* $A_\infty = \{z \in \mathbb{C} \,|\, |z| > 1\}$.
A is open, and the inclusion $A_\infty \subset A$ is proper. We next write A as the disjoint union of its connected components, say $A = A_\infty \cup (\cup_{i \in I} A_i)$, where I is non-empty (and at most countable). Since $0 \notin A$, at least one of the A_i's must be properly contained in the open disk $\mathbb{D}$. But then there exists i_0 such that $\partial A_{i_0} \subset \mathbb{D}$. Thanks to the equality $\partial C = \partial A = \partial A_\infty \cup (\dot{\cup}_{i \in I} \partial A_i)$, we see that ∂C contains points whose absolute value is less than one.
So let $\lambda \in \partial C$ with $|\lambda| < 1$, then $\lambda \in \sigma_{ap}(T)$ by Lemma (2.1), i.e. there exists a sequence $(x_i)_i \subset X$ such that $\|x_i\| = 1$ for all i, and $\|Tx_i - \lambda x_i\| \to 0$ as $i \to \infty$, by definition, but

$$\|Tx_i - \lambda x_i\| \geq \|Tx_i\| - |\lambda| \|x_i\| = \|x_i\| - |\lambda| \|x_i\|,$$

because T is an isometry.
Since $\|x_i\| = 1 \;\; \forall i$ and $|\lambda| < 1$, we find

$$\|Tx_i - \lambda x_i\| \geq 1 - |\lambda| > 0,$$

Thus $\|Tx_i - \lambda x_i\| \nrightarrow 0$ as $i \to \infty$ and we get a contradiction. □

2.3 Scheffold's theorem

As previously mentioned, we include a result stated and proved by Scheffold in [18]. The author basically starts from the following consideration: let X be a compact Hausdorff space and $C(X)$ the Banach space of complex-valued functions on X with usual norm and ordering.
As shown in [22], every lattice homomorphism on $C(X)$ can be represented in the form $Tf(x) = g(x) f(\phi(x))$, for all $x \in X, \; f \in C(X)$, where $g = T\mathbb{1}$ ($\mathbb{1}(x) \equiv 1 \; \forall x \in X$) and ϕ a continuous map between X and itself.
When the function $T\mathbb{1}$ is strictly positive, then the map ϕ - also called the map associated with T - is uniquely determined.

Theorem 2.3.1 (Scheffold). *Let T be a lattice homomorphism on $C(X)$ with $T\mathbb{1} \geq a\mathbb{1}$, $a \in \mathbb{R}, \; a > 0$ and let ϕ the associated map.*

- *If $\phi^{\nu+1}(X) \neq \phi^{\nu}(X)$ for $\nu = 0, 1, 2 \ldots$, then the set $\{\lambda \in \mathbb{C} : |\lambda| < a\} \subset \sigma_{ap}(T)$.*

- *if* $\phi^{M+1}(X) = \phi^M(X)$ *for a non-negative integer* M *then* $\{\lambda \in \mathbb{C} : |\lambda| < a\} \subseteq \sigma_p(T^*)$*, or* $\{\lambda \in \mathbb{C} : 0 < |\lambda| < a\} \cap \sigma(T) = \emptyset$*, with* $0 \in \sigma(T)$ *if* $\phi(X) \neq X$*, and* $0 \notin \sigma(T)$*, if* $\phi(X) = X$*.*

As is clear from the statement, the author chooses to approach the problem using another perspective, that is the case of *Banach lattices.* In fact, as we will see below, the space $C(X)$ is intrinsically multifaceted, and contains within itself a series of mathematical properties of interest. For this reason, we will recall some fundamental results that will come in useful to frame this context.

2.4 Banach lattices

Remark 2.4.1. A partially ordered set $(X, \leq)$ is a lattice if each pair of elements $x, y \in X$ has a supremum and an infimum. An element z is the supremum of a pair of elements $x, y \in X$ if

- z is an upper bound of the set $\{x, y\}$, that is, $x \leq z$ and $y \leq z$,
- z is the least such bound, that is, $x \leq u$ and $y \leq u$ implies $z \leq u$.

The infimum of two elements is defined similarly.

Definition 2.4.1. *An ordered vector space* E *with an order relation* $\leq$ *is a real vector space satisfying the conditions:*

- $x \leq y$ *implies* $x + z \leq y + z \quad \forall x, y, z \in E$*.*
- $0 \leq x$ *implies* $0 \leq tx \quad \forall x \in E$ *and* $t \in \mathbb{R}_+$*.*

An ordered vector space that is also a lattice is called a *Riesz space.*

Definition 2.4.2. *A Banach norm on a vector lattice* E *is called lattice norm, if*

$$|x| \leq |y| \quad \textit{implies} \quad \|x\| \leq \|y\| \quad for\, x\,,\, y \in E.$$

Definition 2.4.3. *A Banach lattice is a real Banach space* E *endowed with an ordering* $\leq$ *such that* $(E, \leq)$ *is a vector lattice and the norm on* E *is a lattice norm.*

Example 2.4.1. $C(X)$, where X is a compact Hausdorff space, endowed with the order

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X$$

is a Banach lattice.

Example 2.4.2. Let $E = C^1[0, 1]$ be the space of continuously differentiable functions in $[0, 1]$. Observe that such space is an ordered vector space under the pointwise ordering, however, it is not a vector lattice.
Indeed, consider the functions $f, g \in E$ given by $f(x) = x$, $g(x) = 1 - x$, and note that both $f \vee g$, $f \wedge g$ are not differentiable at $x = \frac{1}{2}$

Definition 2.4.4. *Given E a normed vector lattice, if the norm on E satisfies*

$$\|x \vee y\| = sup(\|x\|, \|y\|) \quad (x, y \in E_+),$$

then $(E, \|\cdot\|)$ is called an $M-$ normed space, and an $M-$ normed Banach lattice is called an abstract $M-$space, or $AM-$space.
If the norm satisfies

$$\|x + y\| = \|x\| + \|y\| \quad (x, y \in E_+),$$

then $(E, \|\cdot\|)$ is called an $L-$ normed space, and an $L-$ normed Banach lattice is called an abstract $L-$ space, or $AL-$space.

Example 2.4.3. The dual space of a commutative C^*-algebra $\mathfrak{A}$ is an $AL-$space. Indeed, for positive linear functionals $\varphi_1, \varphi_2 \in \mathfrak{A}^*$, one obtains

$$\|\varphi_1 + \varphi_2\| = (\varphi_1 + \varphi_2)(\mathbb{1}) = \varphi_1(\mathbb{1}) + \varphi_2(\mathbb{1}) = \|\varphi_1\| + \|\varphi_2\|,$$

using that $\|\varphi\| = \varphi(\mathbb{1})$. Observe that in this case positive linear functionals are nothing but positive Radon measures. Indeed, the lattice condition is satisfied, as the absolute value $|\mu| = \mu^+ + \mu^-$ is uniquely determined by virtue of the Jordan decomposition of a measure.
In general the dual of a C^*-algebra is not an $AL-$ space, despite respecting the additivity on the cone of positive elements since its dual fails to be a lattice.

Example 2.4.4. Let X be any topological space. The vector space $C_b(X)$ of all bounded real valued continuous functions on X, endowed with its canonical order, is an $AM-$space, with unit $\mathbb{1}$ (where $\mathbb{1}(x) = 1\ \forall x \in X$).

The above example is exactly the prototype of any AM-space.
In fact, a famous result independently obtained by M. Kakutani and S. Krein states the following:

Theorem 2.4.1 (Kakutani). *Let E be an $AM-$ space with unit and denote by K the weak*- compact set of real valued lattice homomorphisms of norm one on E. The evaluation map $x \mapsto f$ (where $f(t) = \langle x, t\rangle$, $t \in K$), establishes an isomorphism between E and $C(K)$.*

Proof. For a detailed proof, the reader is referred to [17]. □

An analogue of the previous theorem, always due to Kakutani, concerns the representation of $AL-$ spaces:

Theorem 2.4.2. *For every $AL-$ space E, there exists a locally compact space X and a strictly positive Radon measure μ on X such that E is isomorphic to $L^1(X, \mu)$.*

Proof. For a detailed proof, the reader is referred to [17]. □

It is surely interesting to understand the underlying relations between $AM-$ and $AL-$ spaces. In particular, they are known to be dual to each other in the following precise sense:

Proposition 2.1. *The dual of any $M-$ normed space is an $AL-$ space, and the dual of any $L-$ normed space is an $AM-$ space with unit.*

Proof. For a detailed proof, the reader is referred to [17]. □

At this point, one could wonder what the relation between C^*- algebras and Banach lattices is. It has been clearly established that if a C^*-algebra $\mathfrak{A}$ is commutative, then the canonical ordering among the self-adjoint elements ($a \leq b \Leftrightarrow a - b \in \mathfrak{A}^+$) is a lattice ordering (this follows from the fact that $C(X)$ is always a lattice under pointwise operations). Furthermore, S. Sherman proved in [19] even the converse:

Theorem 2.4.3. *If $\mathfrak{A}_{sa}$, the set of self-adjoint elements of a C^*-algebra $\mathfrak{A}$ is lattice ordered, then $\mathfrak{A}$ is necessarily commutative.*

Proof. For a detailed proof, the reader is referred to [19]. □

Definition 2.4.5. *If we denote with $E^+ := \{x \in E \; : \; x \geq 0\}$ the cone of positive elements, then a* linear form *μ on E^+ will be* positive *if $\mu(x) \geq 0 \; \forall x \in E^+$ ($\mu \geq 0$).*

Definition 2.4.6. *A linear map $T : E \to E$ is* positive *if the cone of positive elements E^+ is T-invariant ($TE^+ \subset E^+$).*

As is well known, positive linear forms/maps on Banach lattices are automatically continuous:

Proposition 2.2. *Let $E_{\mathbb{C}}$ the complexification of E, so each element $z \in E_{\mathbb{C}}$ is represented as $z = x + iy, \quad x, y \in E$. Furthermore, the immersion of K defines a positive cone , and hence an order in $E_{\mathbb{C}}$.*
Absolute value and norm are now changed with

$$E \ni |z| := sup\{cos(\alpha)x + (sin\alpha)y \; : \; 0 \leq \alpha < 2\pi\} \quad , \quad \|z\| := \||z|\|.$$

With the above norm and absolute value, $E_{\mathbb{C}}$ is a Banach space, and a Banach lattice.

We now move on to define the ideals in a Banach lattice:

Definition 2.4.7. *A subset S of a vector lattice E is called solid if $x \in S$, $|y| \leq |x| \Rightarrow y \in S$. A solid linear subspace is called an ideal.*

Remark 2.4.2. If F is a Banach space, we denote with F^*, F^{**} the dual and bidual spaces, respectively.

Remark 2.4.3. The dual space of a Banach lattice is a Banach lattice and if $E_{\mathbb{C}}$ is a complexified lattice then $E^*_{\mathbb{C}}$ is the complexification of the real Banach lattice E^*.

Definition 2.4.8. *A linear form $\mu \in E^*_{\mathbb{C}}$ is positive, if $\mu(x) \geq 0$ for each positive element of $E_{\mathbb{C}}$.*

2.5 Lattice homomorphisms

Remark 2.5.1. We denote with $B(E)$ the Banach algebra of bounded endomorphisms on a Banach space E. Then, if E is a Banach Lattice, each $T \in B(E)$ admits an extension on $E_{\mathbb{C}}$ that we shall still indicate with T ($T(x+iy) := Tx + iTy$, $x, y \in E$). So, the positive operators on $E_{\mathbb{C}}$ are extensions of positive operators on E.

From the definition of absolute value in $E_{\mathbb{C}}$, it follows that for positive operators T, the following inequality holds
$$|Tz| \leq T|z| \quad \forall z \in E_{\mathbb{C}}.$$
We also note that for positive operators in Banach lattices the spectral radius is included in the spectrum, which is proved below.

Proposition 2.3. *Let T be a positive operator in $B(E)$. Then $r(T) \in \sigma(T)$.*

Proof. By definition of $r(T)$, the family of operators $(R(\lambda, T))_{|\lambda|>r(T)}$ cannot be uniformly bounded in $B(E)$; therefore, there exists a sequence (λ_n) such that $|\lambda_n| \to r(T)$ and $lim_n \|R(\lambda_n, T)x\| = +\infty$ for a suitable $x \in E$. Thanks to the positivity of T, we have

$$|R(\lambda_n, T)x| \leq \sum_{n=0}^{\infty} |\lambda_n|^{-(n+1)} T^n |x| = R(|\lambda_n|, T)|x|,$$

and this shows that $lim_n \|R(|\lambda_n|, T)|x|\| = +\infty$. □

A lattice homomorphism in a complex Banach lattice is simply the extension of a bounded operator on the real underlying lattice E.
Now we finally give a formal definition of it, that will be further useful.

Definition 2.5.1. *Let $T \in B(E_{\mathbb{C}})$. Then T is a lattice homomorphism if it satisfies*

$$|Tz| = T|z| \quad \forall z \in E_{\mathbb{C}}$$

Lemma 2.2. *If E is a Banach space and $F \subset E$ a closed subspace, then $(E/F)^* \simeq F^\circ$, where*
$$F^\circ := \{\mu \in E^* \, : \, \mu(x) = 0 \;\; \forall x \in F\}.$$

Proof. The (isometric) isomorphism is given by the map

$$\begin{aligned} \Phi : F^\circ &\longrightarrow (E/F)^* \\ \mu &\longmapsto \tilde{\mu}([x]) := \mu(x). \end{aligned}$$

The definition is well posed, since $[x] = [y] \Leftrightarrow x - y \in F$ implies $\mu(x-y) = 0$, and hence $\mu(x) = \mu(y)$. Now,

$$\|\tilde{\mu}\| = sup_{\|[x]\| \leq 1} |\tilde{\mu}([x])| = sup_{\|[x]\| \leq 1} |\mu(x)| \leq sup_{\|x\| \leq 1} \|\mu\| \|y\|$$

for some $y \in [x]$. Now $\|[x]\| = inf_{y \in [x]} \|y\|$, so, given $\varepsilon > 0$ we find y such that $\|y\| \leq \|[x]\| + \varepsilon$. Therefore,
$$\|\tilde{\mu}\| \leq \|\mu\|(1+\varepsilon) \Longrightarrow \|\tilde{\mu}\| \leq \|\mu\|,$$
while the converse inequality is trivial. So $\|\mu\| = \|\tilde{\mu}\|$ and the surjectivity is easy. □

Proposition 2.4. *Let E be a Banach space, $T \in B(E)$, F a $T-$ invariant closed subspace of E and $\tilde{T}$ the induced operator on E/F ($\tilde{T}(x+f) := Tx + f\,,\;\; x \in E$).*
If $\alpha \in \sigma_p(T^*)$ with $T^*\mu = \alpha\mu$ and $0 \neq \mu \in F^\circ$, then $\alpha \in \sigma_p(\tilde{T}^*)$, conversely, $\beta \in \sigma_p(\tilde{T}^*)$ always implies $\beta \in \sigma_p(T^*)$.

Proof. With (2.2) at hand, it's enough to note that

$$\begin{array}{c} T : E \to E \Longrightarrow \tilde{T} : E/F \to E/F \\ \Downarrow \\ \tilde{T}^* : F^\circ \to F^\circ. \end{array}$$

The point is that $\tilde{T^*}\tilde{\mu} = T^*\mu$, if $\mu \in F^\circ$. Indeed,

$$\langle \tilde{T}^*\tilde{\mu}, [x]\rangle = \langle \tilde{\mu}, \tilde{T}[x]\rangle = \langle \tilde{\mu}, [Tx]\rangle = \mu(Tx) = \langle T^*\mu, x\rangle.$$

So $T^*\mu = \alpha\mu$ implies $\tilde{T^*}\tilde{\mu} = \alpha\tilde{\mu}$. The converse is obvious. □

Remark 2.5.2. Let E be a real Banach lattice. We now construct for each positive linear form $\nu \in E^*$ a real Banach lattice F_ν in the following way: on E we consider the seminorm $p_\nu(x) := \nu(|x|)\ \forall x \in E$; p_ν is a continuous seminorm on E. Let now $I_\nu := \{x \in E : p_\nu(x) = 0\}$, so I_ν is a lattice ideal of E. It is known that the quotient space E/I_ν, obtained in the usual way, is a vector lattice, and p_ν induces in E/I_ν a lattice norm $\tilde{p}_\nu([x]) := p_\nu(x)$ $(x \in [x])$ $\forall [x] \in E/I_\nu$. This comes from the fact that the canonical map $x \mapsto [x]$ of E into E/I_ν is a lattice homomorphism. Now let F_ν be the completion of vector lattice E/I_ν with the norm $\tilde{p}_\nu$. Then F_ν is a real Banach lattice and also a $AL-$ space, since the norm is additive on the cone of positive elements. By the Kakutani representation theorem, F_ν as Banach lattice is isomorphic to some $L^1(\mu)$.

Proposition 2.5. *Let E be a real Banach lattice and ν a positive linear form on E^*. Then the dual $(F_\nu)^*$ of F_ν is isomorphic to the sub-lattice*

$$E^*_\nu := \bigcup_{n \in \mathbb{N}} n\{\mu \in E^* \;:\; |\mu| \leq \nu\} \qquad \textit{in } E^*.$$

Proof. As above, let $F_\nu = \widetilde{E/I_\nu}$ denote the completion of the quotient space E/I_ν, and consider

$$F^*_\nu = \{\mu \in E^* \;:\; |\mu| \leq c\nu \quad \textit{for some } c > 0\},$$

First of all, we observe that if $\mu \in E^*$ is such that $|\mu| \leq c\nu$ then $\nu(|x|) = 0 \Rightarrow \mu(x) = 0$. Now $\mu \leq |\mu|$, so $\mu(x) \leq |\mu|(x) \leq c\nu(x) \leq c\nu(|x|) = 0$, and hence $\mu(x) \leq 0$. Similarly, $-\mu(x) = \mu(-x) \leq |\mu|(-x) \leq c\nu(-x) \leq c\nu(|x|) = 0$, from which $\mu(x) \geq 0$ and hence $\mu(x) = 0$.
Of course μ extends to the completion $F_\nu = \widetilde{E/I_\nu}$, since it is a bounded functional, namely $|\mu([x])| \leq cp_\nu([x])$.
Conversely, let $\varphi \in F^*_\nu$ and define

$$\begin{aligned} &\mu : E \longrightarrow \mathbb{R} \\ &\mu(x) := \varphi[x], \qquad [x] \in E/I_\nu. \end{aligned}$$

Then $|\mu| \leq \|\varphi\|\nu$. Indeed,

$$\mu(x) = \varphi([x]) \leq |\varphi([x])| \leq \|\varphi\| p_\nu(|x|) = \|\varphi\|\nu(|x|), \qquad (\,\|[x]\| = p_\nu(|x|)\,).$$

So $\mu(x) \leq \|\varphi\|\nu(x)$ for all $x \geq 0$, and similarly,

$$-\mu(x) = \mu(-x) = \varphi([-x]) \leq \|\varphi\| p_\nu(x) = \|\varphi\|\nu(|x|).$$

Namely, $-\mu(x) \leq \|\varphi\|\nu(x) \quad \forall x \geq 0$.

$$|\mu| \leq \|\varphi\|\nu.$$

□

Proposition 2.6. *Let T be a positive operator in a real Banach lattice E and $\nu \in E^*$ a positive linear form.*
If $T^*\nu \leq r\nu$ for some $0 < r \in \mathbb{R}$ and $0 < \nu \in E^*$, then T induces canonically a positive operator $\tilde{T}$ in the AL− Space F_ν.
For $\alpha \in \sigma_p(T^*)$ with $T^*\mu = \alpha\mu$ and $0 \neq |\mu| \in (E^*)_\nu$ it holds $\alpha \in \sigma_p(\tilde{T}^*)$. Conversely, $\beta \in \sigma_p(\tilde{T}^*)$ implies $\beta \in \sigma(T^*)$.

Proof. From $T^*\nu \leq r\nu$, it follows $T(I_\nu) \subseteq I_\nu$, i.e. I_ν is $T-$invariant, which allows one to define an operator $\tilde{T}$ on E/I_ν as $\tilde{T}[x] := [Tx]$ $(x \in [x])$, for every $[x] \in E/I_\nu$. $\tilde{T}$ is then easily seen to extend a positive operator on F_ν. In light of proposition (2.5) $\tilde{T}^*$ can now be identified with the restriction of T^* to E^*_ν. From this, the statement about the point spectrum (eigenvalues) of T^* follows immediately. □

We will only need the first part of the next theorem, but for the sake of completeness, we include the following definition:

Definition 2.5.2. *A subset $A \subset \mathbb{C}$ is said to be cyclic if $\alpha \in A$, $\alpha = |\alpha|\gamma$ implies $|\alpha|\gamma^k \in A$ $\forall k \in \mathbb{Z}$.*

Theorem 2.5.1. *Let T be a lattice homomorphism in Banach lattice E. If $\alpha \neq 0$ is an eigenvalue of the adjoint T^* with $\alpha = |\alpha|\beta$, then the set $\{\lambda \in \mathbb{C} : |\lambda| < |\alpha|\}$ is contained in the point spectrum of the adjoint $\sigma_p(T^*)$, or it holds $|\alpha|\beta^k \in \sigma_p(T^*)$ $\forall k \in \mathbb{Z}$.*

Proof. Let $\mu \in E^*$ be an eigenvector of T^* whose eigenvalue is $\alpha \neq 0$ and $\nu := T^*R(r_1, T^*)|\mu|$, with $r_1 > r(T)$. The following relations come from $T^*|\mu| \geq |\alpha||\mu|$ and from the definition of ν:

1. $0 < |\alpha|\nu \leq T^*\nu \leq r_1\nu$,
2. $|\alpha||\mu| \leq T^*|\mu| \leq r_1\nu$.

According to 1., T induces an operator $\tilde{T}$ on the Banach lattice F_ν by (2.6), and $\tilde{T}$ is a lattice operator too. From 1. we derive also the relation

3. $\|\tilde{T}y\| \geq |\alpha|\|y\| \ \forall y \in F_\nu, \quad \|y\| := \tilde{p}_\nu(y), \quad (p_\nu(x) := \nu(|x|))$

But this is equivalent to saying $|\gamma| \geq \alpha \quad \forall \gamma \in \sigma_{ap}(\tilde{T})$.
Now, if $0 \in \sigma(\tilde{T})$, then

$$\{\lambda \in \mathbb{C} \,:\, |\lambda| < |\alpha|\} \subset \sigma(\tilde{T}),$$

because boundary of $\sigma(\tilde{T})$ lies in $\sigma_{ap}(\tilde{T})$. Furthermore, it will hold

$$\{\lambda \in \mathbb{C} \,:\, |\lambda| < |\alpha|\} \subset \sigma_p(T^*),$$

since it is a known fact that $\sigma(\tilde{T}) = \sigma_{ap}(\tilde{T}) \cup \sigma_p(\tilde{T}^*)$, and from inequality 3. it turns out that

$$\sigma_{ap}(\tilde{T}) \cap \{\lambda \in \mathbb{C} \,:\, |\lambda| < |\alpha|\} = \emptyset.$$

If otherwise $0 \notin \sigma(\tilde{T})$, then both $\tilde{T}$ and $\tilde{T}^*$ are Banach isomorphisms.
Summing up, the statement of the theorem is the following:

- For $0 \in \sigma(\tilde{T})$, then $\{\lambda \in \mathbb{C} \,:\, |\lambda| < |\alpha|\} \subset \sigma_p(\tilde{T}^*)$ and by (2.6) it holds $\{\lambda \in \mathbb{C} \,:\, |\lambda| < |\alpha|\} \subset \sigma_p(T^*)$.
- For $0 \notin \sigma(\tilde{T})$, $\tilde{T}^*$ is a lattice operator. From inequality 2. , it follows $|\mu| \in E^*_\nu$. By (2.6), α is an eigenvalue of $\tilde{T}^*$. So, by ([12], 3.2), $|\alpha|\beta^k \in \sigma_p(\tilde{T}^*)$ $\forall k \in \mathbb{Z}$. By (2.6) it holds $|\alpha|\beta^k \in \sigma_p(T^*)$ $\forall k \in \mathbb{Z}$.

□

With these tools at hand, we finally provide the proof of the main theorem of this chapter:

Proof. of Theorem (2.3.1)

- For each $n \in \mathbb{N}$ there exists a point $x_n \in X$ with $\phi^n(x_n) \notin \phi^{n+1}(X)$ and $\phi^\nu(x_n) \neq \phi^\mu(x_n)$ for $\nu \neq \mu$, ν, $\mu = 0, 1, 2 \ldots n$. Moreover, *Urysohn lemma* gives us a function $f_n \in C(X)$ with $f_n(\phi^\nu(x_n)) = 0$, $(\nu = 0, 1, \ldots, n-1)$, $f_n(\phi^n(x_n)) = 1$, $f_n \equiv 0$ on $\phi^{n+1}(X)$ and $\|f_n\| = 1$.
  Let $|\lambda| < a$ and $g_n := \sum_{\nu=0}^n \lambda^\nu T^{n-\nu} f_n$, it holds $g_n(x_n) \geq a^n$ (i.e. $\|g_n\| \geq a^n$) and $Tg_n - \lambda g_n = -\lambda^{n+1} f_n$. Furthermore, by defining $h_n := \|g_n\|^{-1} g_n$, one obtains $\|Th_n - \lambda h_n\| = |\lambda|^{n+1}\|g_n\|^{-1} \leq |\lambda|^{n+1} a^{-n}$. The sequence $\{(Th_n - \lambda h_n)\}_{n \in \mathbb{N}}$ is null with $\|h_n\| = 1$, that is, $\lambda \in \sigma_{ap}(T)$.
- Let m be the smallest non-zero integer such that $\phi^{m+1}(X) = \phi^m(X)$, and let also $Y = \phi^m(X)$ and $J = \{f \in C(X) \,:\, f \equiv 0 \quad on \quad Y\}$. The latter is a closed $T-$ invariant lattice ideal of $C(X)$ and the quotient space $C(X)/J$ can be identified with $C(Y)$.
  Moreover, T induces a lattice operator $\tilde{T}$ with $\|\tilde{T}[f]\| \geq a\|[f]\|$ for all $[f] \in C(Y)$. If $0 \in \sigma(\tilde{T})$, by the previous norm relation the set $\{\lambda \in \mathbb{C} \,:\, |\lambda| < a\}$ is contained in the point spectrum of $\tilde{T}^*$ and by (2.4), in the point spectrum of T^*. Otherwise, if $0 \notin \sigma(T^*)$, then $\{\lambda \in \mathbb{C} \,:\, |\lambda| < a\} \cap \sigma(\tilde{T}) = \emptyset$. If we indicate with T_1 the restriction of T on the subspace J, then T_1 is nilpotent and $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(\tilde{T})$. In this case, it follows that $\sigma(T) \cap \{\lambda \in \mathbb{C} \,:\, 0 < |\lambda| < a\} = \emptyset$.
  If $m = 0$, then $J = \{0\}$, $T = \tilde{T}$ and $\sigma(T) \cap \{\lambda \in \mathbb{C} \,:\, |\lambda| < a\} = \emptyset$. If $m \geq 1$, then $0 \in \sigma(T)$.

□

As a possible problem we would like to face in the foreseeable future, it might be worth asking whether suitable generalizations of the above result can be given in the wider setting of (infinite-dimensional) non-commutative C^*-algebras.
For the time being, we can point out a couple of remarks, the first of which is very general in character. Although it might be well known to experts, we do include it for want of a precise reference.

Remark 2.5.3. If $\mathfrak{A}$ is a C^*-algebra and $\Phi \in Aut(\mathfrak{A})$, then it results $\|\Phi\| = 1$ and therefore $\sigma(\Phi) \subset \overline{\mathbb{D}}$. But in particular, since Φ is invertible we have $0 \notin \sigma(\Phi)$. Furthermore, if there was $\lambda \in \sigma(\Phi)$ with $|\lambda| < 1$ there would exist $\mu \in \sigma(\Phi^{-1})$ with $|\mu| > 1$ given that $\sigma(\Phi^{-1}) = \frac{1}{\sigma(\Phi)}$ and this would result in $\|\Phi\| \geq r(\Phi^{-1}) > 1$, but this is clearly impossible, as Φ^{-1} is in turn a $^*-$ automorphism and therefore $\|\Phi^{-1}\| = 1$.

With the previous remark at hand we are in a position to fully settle the problem for a *simple* C^*-algebra $\mathfrak{A}$. Indeed, in this case there are no non-trivial two-sided ideals, which means any *-endomorphims Φ of $\mathfrak{A}$ is automatically injective.
If $\mathfrak{A}$ is finite-dimensional, namely $\mathfrak{A} \simeq M_n(\mathbb{C})$, for some $n \in \mathbb{N}$, then Φ is an automorphism, which means its spectrum is a finite subset of $\mathbb{T}$. If $\mathfrak{A}$ is not finite-dimensional, there are two casese than can occur.
First, let us recall that injective *-endomorphsims are known to be isometric. Now either Φ fails to be surjective, in which case its spectrum is full, i.e. $\sigma(\Phi) = \overline{\mathbb{D}}$, by virtue of Theorem 2.2.1 or Φ is an automorphism, in which case $\sigma(\Phi) \subset \mathbb{T}$ thanks to Remark 2.5.3.

As well as these general observations, we can also give a full treatement of the problem for *-endomorphisms of finite-dimensional C^*-algebras, which we do in the next section.

2.6 The finite-dimensional case

The main goal of this section is to show the spectrum of any *-endomorphism of a finite-dimensional C^*-algebra, which is of course a finite set of the unit closed disk $\mathbb{D}$, can only have 0 has a value that does not sit in its peripheral spectrum. To be sure, the statement is not easily found in the literature and, as far as we know, the result is likely to be new, which is why we provide a thorough proof.

The first thing we would like to do is to recall what finite-dimensional C^*-algebras look like.

Theorem 2.6.1. *Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra. Then $\mathfrak{A}$ is isomorphic with a direct sum of full matrix algebras, i.e. there exist positive integers $m_1, m_2, \dots m_r$ such that*

$$\mathfrak{A} \simeq M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \cdots \oplus M_{m_r}(\mathbb{C}). \tag{2.2}$$

Moreover, if

$$M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \cdots \oplus M_{m_r}(\mathbb{C}) \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \cdots \oplus M_{n_r}(\mathbb{C}),$$

then $r = n$ and there exists a permutation σ of $\{1, 2, \dots, r\}$ such that $n_i = m_{\sigma(i)}$.

Remark 2.6.1. If in addition our C^*-algebra $\mathfrak{A}$ is commutative, then $\mathfrak{A}$ is isomorphic with $\mathbb{C}^n$, for some $n \in \mathbb{N}$. This is a straightforward application of the theorem above. However, we can leap to the same conclusion as easily by applying the Gelfand-Naimark theorem instead. Indeed, in this case the spectrum X of $\mathfrak{A}$ is necessarily a finite set, say $X = \{1, 2, \dots, n\}$, which means $\mathfrak{A} \simeq C(X) \simeq \mathbb{C}^n$.

We first tackle the commutative case, to which we then reconduct the general case. To do so, we need an algebraic result known as Kronecker's theorem, which we recall below for convenience.

Theorem 2.6.2 (Kronecker). *Every non-zero algebraic integer that lies with its conjugates in the closed unit disk $\overline{\mathbb{D}}$ is a unit root.*

Proof. For a detailed proof, the reader is referred to [11]. □

We can now move on to the commutative case.

Lemma 2.3. *Let Φ be a *-endomorphism of $\mathbb{C}^n$. If $\lambda \in \sigma(\Phi)$, then either $\lambda = 0$ or $|\lambda| = 1$ and λ is a root of unity.*

Proof. Let $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{C}^n$. In terms of functions, e_i is the characteristic function of the singleton $\{i\}$. By Gelfand duality Φ is induced by a map acting at the spectrum level, namely there exists a (not necessarily injective) function f from $\{1, 2, \dots, n\}$ to itself such that $\Phi(e_i) = e_i \circ f$. Now if f were injective, we could simply write $\Phi(e_i) = e_{f^{-1}(i)}$. The case when f fails to be injective can be treated by passing to the dual. Let $\{e_i^*\}$ be the dual canonical basis and Φ^t the transposed map acting on the dual $\mathfrak{A} \simeq \mathbb{C}^n$. Now the equality $\Phi^t e_i^* = e_{f(i)}^*$ is easily seen to hold. Therefore, in the dual canonical basis Φ^t is represented by a matrix A whose rows have only one 1 and the other entries are 0. But then the characteristic polynomial p of such a matrix is a monic polynomial whose coefficients are all integer numbers, i.e. $p(x) \in \mathbb{Z}[x]$. Because an eigenvalue λ of Φ^t is a root of $p(x)$, we see at once that either $\lambda = 0$ or $|\lambda| = 1$ by applying Kronecker's theorem, and the conclusion is got to since $\sigma(\Phi) = \sigma(\Phi^t)$. □

Remark 2.6.2. The above result holds for finite-dimensional C^*-algebras only. A very simple infinite-dimensional counterexample is provided by $\mathfrak{A} = C[0,1]$ and $\Phi \in \mathrm{End}(\mathfrak{A})$ given by $\Phi(f)(x) := f(\frac{x}{2})$, for every $x \in [0,1]$. Indeed, the infinite sequence $\{\frac{1}{2^n} : n \in \mathbb{N}\}$ is all contained in the point spectrum of Φ because $\Phi(f_n) = \frac{1}{2^n} f_n$ if $f_n(x) = x^n$, for every $n \in \mathbb{N}$.

Remark 2.6.3. A purely C^*-algebraic proof of the above lemma, which is moreover far more straighforward, can also be given. Indeed, if λ is an eigenvalue of Φ with $\Phi(a) = \lambda a$, where $a \in \mathfrak{A}$ is not null, then $\Phi(a^n) = \lambda^n a^n$. Because $\mathfrak{A}$ is commutative, each a^n is different from zero. Therefore, the sequence $\{\lambda^n : n \in \mathbb{N}\}$ is all contained in the spectrum of Φ. Since the latter is a finite set, the sequence $\{\lambda^n : n \in \mathbb{N}\}$ must be finite as well. Now there are only two possibilities for the sequence to be a finite set: either $\lambda = 0$ or $\lambda^{n_0} = 1$ for some integer n_0, in which case $|\lambda| = 1$.

We are now in a position to prove the main theorem of this section.

Theorem 2.6.3. *Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra and $\Phi \in End(\mathfrak{A})$. Then*

$$\lambda \in \sigma(\Phi) \Longrightarrow |\lambda| = 1 \quad or \quad \lambda = 0. \tag{2.3}$$

Proof. Let λ be an eigenvalue of $\mathfrak{A}$ and let $a \in \mathfrak{A} \setminus \{0\}$ be a corresponding eigenvector, that is $\Phi(a) = \lambda a$. Since Φ is a *-endomorphism, we also have the equality $\Phi(a^*a) = \Phi(a)^*\Phi(a) = |\lambda|^2 a^*a$. If we set $b := a^*a$, then b is positive and $\Phi(b) = |\lambda|^2 b$. In other words, b is a positive eigenvector associated with the positive eigenvalue $|\lambda|^2$. Now let us consider the sub-C^*-algebra $\mathfrak{B} \subset \mathfrak{A}$ generated by b, i.e. $\mathfrak{B} = \{p(b) : p(x) \in \mathbb{C}[x]\}$. We have:

1. As b is self-adjoint, $\mathfrak{B}$ is commutative, i.e. $\mathfrak{B} \simeq \mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

2. $\mathfrak{B}$ is Φ-invariant. Indeed, $\Phi(p(b)) = p(\Phi(b)) = p(|\lambda|^2 b)$ where p is any polynomial.

But then Lemma 2.3 applies and $|\lambda|^2$ is either 0 or 1, which ends the proof.

□

Chapter3

On the structure of the persistent part of a C^*-dynamical system

In this chapter we discuss and expand in some detail those considerations that at the beginning were the prime motivation for the whole work, along with a few results and examples we have been able to come up with thus far.

3.1 The persistent and transient part of a mass-gapped map Φ

The next result shows how to define two projections, say Q and P, associated with any unital completely positive map Φ, under the hypothesis that the spectrum of Φ displays a mass gap, namely when its peripheral spectrum $\sigma_{ph}(\Phi) := \sigma(\Phi) \cap \mathbb{T}$ is at a non-zero distance from $\sigma(\Phi) \setminus \sigma_{ph}(\Phi)$. As will be clear, Q is the projection onto the transient part, whereas $P = 1 - Q$ is the projection onto the persistent part.

Proposition 3.1. *Let $\mathfrak{A}$ be a unital C^*-algebra and Φ an identity-preserving positive map. We suppose that*

$$dist(\sigma_{ph}(\Phi)\,,\,\sigma(\Phi) \setminus \sigma_{ph}(\Phi)) > 0. \quad (mass\ gap\ for\ \sigma(\Phi))$$

that is, there is a curve γ_r $(0 < r < 1)$ separating σ_{ph} from $\sigma(\Phi) \setminus \sigma_{ph}(\Phi)$, where σ_{ph} denotes the peripheral spectrum of Φ, defined as

$$\sigma_{ph}(\Phi) := \sigma(\Phi) \cap \mathbb{T}. \tag{3.1}$$

If $Q(x) := \frac{1}{2\pi i} \oint_{\gamma_r} R(z, \Phi)x\, dz$ and $P(x) := I - Q(x)$ is the projection onto the peripheral spectrum of Φ, then $Ran(P)$ is an operator system.

Proof. $\mathbb{1} \in \mathcal{S}$, because Φ is identity preserving, and $\mathcal{S}$ is norm closed since P is continuous. So it is enough to show that $\mathcal{S}$ is self-adjoint, that is $\mathcal{S} = \mathcal{S}^*$:
one has

$$Q(x)^* = \left(\frac{1}{2\pi i} \oint_{\gamma_r} R(z, \Phi)xdz \right)^*$$

and

$$\frac{1}{2\pi i}\oint_{\gamma_r} R(z,\Phi)x\,dz = \frac{\rho}{2\pi}\int_0^{2\pi} R(\rho e^{i\theta},\Phi)xe^{i\theta}d\theta.$$

Since $(R(z,\Phi)(x))^* = R(\overline{z},\Phi)(x^*)$ (Φ preserves the $^*-$operation), then

$$\begin{aligned}\left(\int_0^{2\pi} R(\rho e^{i\theta},\Phi)xe^{i\theta}\frac{d\theta}{2\pi}\right)^* &= \int_0^{2\pi} R(\rho e^{-i\theta},\Phi)x^*e^{-i\theta}\frac{d\theta}{2\pi}\\ &= -\int_{2\pi}^{0} R(\rho e^{i\alpha},\Phi)x^*e^{i\alpha}\frac{d\alpha}{2\pi},\end{aligned}$$

where $\theta = -\alpha$, $d\theta = -d\alpha$. Therefore,

$$\left(\frac{1}{2\pi i}\oint_{\gamma_r} R(z,\Phi)x\,dz\right)^* = \frac{1}{2\pi i}\oint_{\gamma_r} R(\overline{z},\Phi)x^*\,dz,$$

and hence

$$(Px)^* = x^* - \frac{1}{2\pi i}\oint_{\gamma_r} R(z,\Phi)x^*\,dz,$$

which means

$$(Px)^* = Px^*. \tag{3.2}$$

□

Remark 3.1.1. The hypothesis on the mass gap is clearly satisfied in all *finite-dimensional* examples, for in this case the spectrum of Φ is even a finite subset of the closed disk.

Assuming that we do have a mass gap, we are able to separate the transient from the persistent part. In particular, the transient part of the system will correspond to the range of the projection Q onto the part of the spectrum delimited by the curve γ_r, as the next theorem shows:

Theorem 3.1.1. *Let $(\mathfrak{A},\Phi)$ a C^*-system, where Φ is a positive map. Then, in the same hypotheses of (3.1), we have*

$$\lim_{n\to\infty} \|\Phi^n \circ Q\| = 0. \tag{3.3}$$

Proof. We start by showing the following identity, which holds for any natural number n:

$$\Phi^n \circ Q = \frac{1}{2\pi i}\int_{\gamma_r} z^n R(z,\Phi)\,dz. \tag{3.4}$$

We will argue by induction on n. If $n = 1$, by linearity we have

$$\Phi \circ Q = \frac{1}{2\pi i}\int_{\gamma_r} \Phi R(z,\Phi)\,dz$$

If we now make use of the known identity $\Phi R(z,\Phi) = zR(z,\Phi) - \mathbb{1}$, we can rewrite the above formula as

$$\Phi \circ Q = \frac{1}{2\pi i}\int_{\gamma_r} (zR(z,\Phi) - 1)\,dz = \frac{1}{2\pi i}\int_{\gamma_r} zR(z,\Phi)\,dz,$$

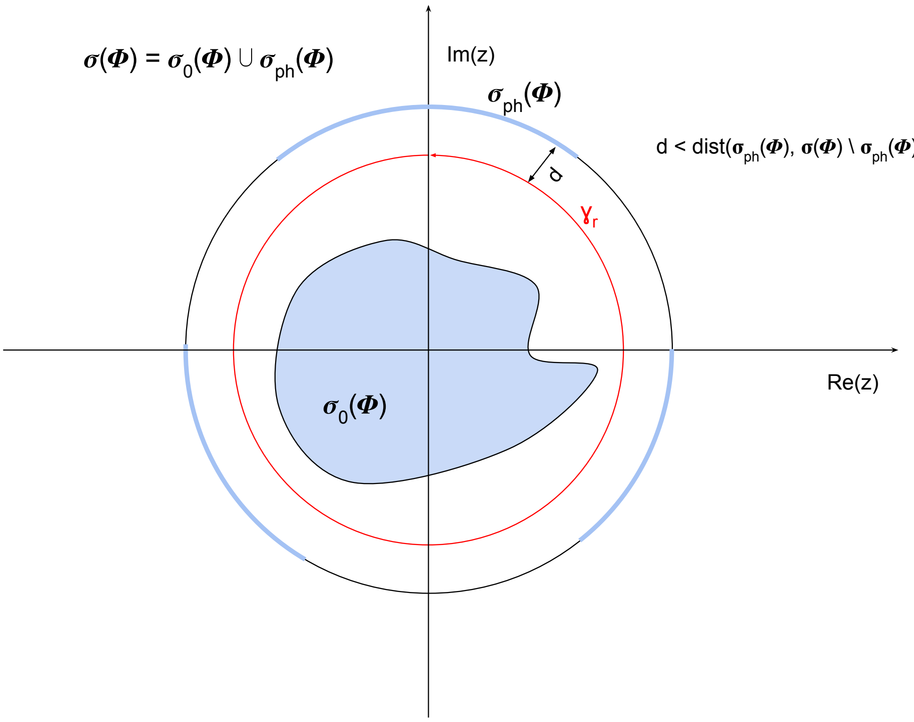

Figure 3.1: The figure shows an example of how the spectrum might look like in presence of a mass gap.

where the second equality follows from Cauchy's theorem. The inductive step can be performed in much the same way. Indeed, we have:

$$\begin{aligned}\Phi^{n+1} \circ Q = \Phi \circ (\Phi^n \circ Q) &= \frac{1}{2\pi i}\int_{\gamma_r} z^n \Phi R(z,\Phi)\, dz = \frac{1}{2\pi i}\int_{\gamma_r} (z^{n+1}R(z,\Phi) - z^n \mathbb{1})\, dz \\ &= \frac{1}{2\pi i}\int_{\gamma_r} z^{n+1}R(z,\Phi) dz.\end{aligned}$$

where the last equality is again due to Cauchy's formula. The conclusion can now be easily achieved, for we have the following estimate:

$$\|\Phi^n \circ Q\| \leq \frac{1}{2\pi}\int_{\gamma_r} \|z^n R(z,\Phi)\|\, dz \leq \frac{1}{2\pi}|z|^n 2\pi r \max_{|z|=r} \|R(z,\Phi)\| = |r|^{n+1} M \xrightarrow[n\to\infty]{} 0.$$

where M is the maximum of the norm of the resolvent of Φ along the curve γ_r. □

Remark 3.1.2. In the proof of the above proposition positivity actually plays no role, and the result continues to hold even if Φ is only assumed to be a bounded linar map of norm 1 provided that its spectrum does display a mass gap.

We would like to end this section by pointing out what we believe is an interesting open problem, namely if it is still possible to separate the persistent from the transient part under the hypothesis that the spectrum is the whole closed unit disk, and therefore without assuming the presence of a mass-gap.

3.2 Injective operator systems

Going back to the operator system $\mathcal{S}$, our ultimate goal would be to see whether it can also be endowed with a C^*-algebra structure as well. Unfortunately, we do not have a full answer yet.
Our educated guess, though, is that far more could be said when the projection P is a completely positive map. Indeed, in that case a natural way to proceed would be to reproduce the proof of a well-known result by Choi and Effros that *injective* operator systems can always be endowed with a C^*-algebra structure.
Because we believe this strategy is really worth attempting, we include a brief treatment of injectivity aimed at highlighting the analogy with our context.

First, we recall what injective means. Roughly speaking, an injective operator system is one for which a suitable version of the Hanh-Banach theorem holds for completely positive maps rather than mere bounded linear maps.

Definition 3.2.1. *An operator system $\mathcal{S} \subseteq B(\mathcal{H})$ is called injective, if given an operator system inclusion $\mathcal{N} \subseteq \mathcal{M} \subseteq B(\mathcal{K})$, any completely positive map $\phi : N \to S$ has an extension $\phi : M \to \mathcal{S}$ that is still completely positive.*

Remark 3.2.1. $B(\mathcal{H})$ is an injective operator system, as shown in the following famous theorem proved by Arveson:

Theorem 3.2.1 (Arveson's extension theorem). *Let $\mathfrak{A}$ be a C^*- algebra, $\mathcal{S}$ an operator system contained in $\mathfrak{A}$, and $\phi : \mathcal{S} \to B(\mathcal{H})$ a completely positive map. Then there exists a completely positive map $\psi : \mathfrak{A} \to B(\mathcal{H})$ extending ϕ.*

Proof. Let $\mathcal{F}$ a finite-dimensional subspace of $\mathcal{H}$, and let $\phi_{\mathcal{F}} : \mathcal{S} \to B(\mathcal{F})$ be the compression of Φ to $\mathcal{F}$, i.e. $\Phi_{\mathcal{F}}(a) = P_{\mathcal{F}}\Phi(a)|_{\mathcal{F}}$, where $P_{\mathcal{F}}$ is the projection onto $\mathcal{F}$. Since $B(\mathcal{F})$ is isomorphic to M_n for some n, by Theorem 6.2 of [14] there exists a completely positive map $\psi_{\mathcal{F}} : \mathfrak{A} \to B(\mathcal{F})$ extending $\Phi_{\mathcal{F}}$.
Let $\psi'_{\mathcal{F}} : \mathfrak{A} \to B(\mathcal{H})$ be defined by setting $\psi'_{\mathcal{F}}(a)$ equal to $\psi_{\mathcal{F}}(a)$ on $\mathcal{F}$ and extending it to be 0 on $\mathcal{F}^{\perp}$. The set of finite-dimensional subspaces of $\mathcal{H}$ is a directed set under inclusion, and so $\{\psi'_{\mathcal{F}}\}$ is a net in $CP_r(\mathfrak{A}, \mathcal{H})$, where

$$CP_r(\mathfrak{A}, \mathcal{H}) = \{L \in B(\mathfrak{A}, B(\mathcal{H})) \,:\, L \textit{ is completely positive }, \|L\| \leq r\},$$

and $r = \|\Phi\|$.
Since the latter is a compact set, we may choose a subnet which converges to some element $\psi \in CP_r(\mathfrak{A}, \mathcal{H})$. We claim that ψ is the desired extension. Indeed, if $a \in \mathcal{S}$ and x, y are in $\mathcal{H}$, let $\mathcal{F}$ be the space spanned by x, y. Then, for any $\mathcal{F}_1 \supseteq \mathcal{F}$, $\langle \Phi(a)x, y\rangle = \langle \psi_{\mathcal{F}_1}(a)x, y\rangle$, and since the set of such $\mathcal{F}_1$ is cofinal, we have that $\langle \Phi(a)x, y\rangle = \langle \psi(a)x, y\rangle$.
This completes the proof of the theorem. □

Obviously, $B(\mathcal{H})$ is much more than a mere operator system as it is even a C^*-algebra. This circumstance is actually no coincidence. In fact, the celebrated theorem of Choi and Effros we have alluded to, see [5], says that in any injective operator system a product between its elements can be defined that turns it into a C^*-algebra.

Theorem 3.2.2 (**Choi-Effros**). *If $\mathcal{S}$ is an injective operator system, then there is an identity preserving isomorphism of $\mathcal{S}$ into a unique (up to $^*-$ isomorphisms) C^*-algebra.*

Before moving to the proof, we first need to recall a preliminary result, which is interesting in its own:

Proposition 3.2 (**Kadison-Schwarz inequality**). *If $\mathfrak{A}$ is a unital C^*-algebra, $B(\mathcal{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and $\Phi : \mathfrak{A} \to B(\mathcal{H})$ a positive unital linear map, then $\Phi(a^2) \geq \Phi(a)^2$ for each self-adjoint element a.*

Proof. For a detailed proof, the reader is referred to [10]. □

Now we give the proof of Choi-Effros:

Proof. Supposing that $\mathcal{S}$ is injective, then the inclusion map $\mathcal{S} \to B(\mathcal{H})$ extends to a completely positive projection P of $B(\mathcal{H})$ onto $\mathcal{S}$. It immediately follows from the continuity of P that $\mathcal{S}$ is norm complete. We claim that

$$P(rP(b)) = P(rb) \quad \forall r \in \mathcal{S}\,,\, b \in B(\mathcal{H}). \tag{3.5}$$

By linearity, we may assume that r and b are self-adjoint, and thus that

$$d = \begin{bmatrix} 0 & r \\ r & b \end{bmatrix} \in M_2(B(\mathcal{H}))_{sa}.$$

Since P_2 is positive, we have from (3.2) that $P_2(d^2) \geq P_2(d)^2$ i.e., since $P(r) = r$,

$$\begin{bmatrix} P(r^2) & P(rb) \\ P(rb) & * \end{bmatrix} \geq \begin{bmatrix} r^2 & r(Pb) \\ P(b)r & * \end{bmatrix},$$

where there is no need to evaluate the omitted entries. Applying P_2 to both sides, we have

$$\begin{bmatrix} P(r^2) & P(rb) \\ P(br) & * \end{bmatrix} \geq \begin{bmatrix} P(r^2) & P(r(Pb)) \\ P(P(b)r) & * \end{bmatrix}.$$

Letting $c = P(rb) - P(rP(b))$, it follows that

$$u = \begin{bmatrix} 0 & c \\ c^* & * \end{bmatrix} \geq 0,$$

and this implies $c = 0$. Indeed, taking $u = v^2$ where

$$v = \begin{bmatrix} v_{11} & v_{12} \\ v_{12}^* & v_{22} \end{bmatrix} \in M_2(B(\mathcal{H}))_{sa},$$

then

$$u = \begin{bmatrix} v_{11}^2 + v_{12}v_{12}^* & v_{11}v_{12} + v_{12}v_{22} \\ v_{12}^*v_{11} + v_{22}v_{12}^* & v_{12}^*v_{12} + v_{22}^*, \end{bmatrix}$$

and hence $v_{11} = v_{12} = 0$. Thus $c = 0$ and we have (3.5). Taking adjoints we also have

$$P(P(b)r) = P(br). \tag{3.6}$$

We define a bilinear product in $\mathcal{S}$ by

$$r \circ s := P(rs). \tag{3.7}$$

This is associative, since by (3.5) and (3.6)

$$P(rP(st)) = P(rst) = P(P(rs)t).$$

We let $\mathcal{S}$ have the relative $*-$ operation and norm, then, since P preserves the $^*-$ operation,

$$(r \circ s)^* = P(rs)^* = P(s^*r^*) = s^* \circ r^*,$$

i.e $\mathcal{S}$ is a $^*-$ algebra. Since P is a contraction (this follows from (1.3.2)),

$$\|r \circ s\| = \|P(rs)\| \leq \|rs\| \leq \|r\|\|s\|.$$

On the other hand, again the Stinespring decompostion for P (1.3.2) implies the following variant of the Kadison-Schwarz inequality

$$r^* \circ r = P(r^*r) \geq P(r)^*P(r) = r^*r,$$

hence

$$\|r \circ r\| \geq \|r^*r\| = \|r\|^2,$$

and $(\mathcal{S}, \circ)$ is a C^*-algebra.
Denoting the product on the C^*-algebra $M_n(\mathcal{S}, \circ)$ again by "$\circ$", we have

$$\begin{aligned}(r_{ij}) \circ (s_{ij}) &= \left(\sum_{k=1}^{n} r_{ik}s_{kj}\right) \\ &= \left(P\left(\sum_{k=1}^{n} r_{ik}s_{kj}\right)\right) \\ &= P_n((r_{ij})(s_{ij})).\end{aligned}$$

This coincides with the product of the C^*-algebra determined as above by the projection $P_n : B(\mathcal{H}^n) \to M_n(\mathcal{S})$. Hence we may identify these two C^*-algebras and in particular $M_n(\mathcal{S}, \circ)$ has the relative $B(\mathcal{H}^n)$ norm. It follows that

$$M_n(\mathcal{S}) \cap B(\mathcal{H}^n)^+ = M_n(\mathcal{S}, \circ)^+,$$

since for a self-adjoint operator a in any C^*-algebra, a is positive if and only if

$$\|(\|a\|\mathbb{1} - a)\| \leq \|a\|.$$

The (essential) uniqueness of such an isomorphism is described in [5]. $\square$

3.3 A class of working examples

At the moment, though, we do not know under what conditions the projection P may be completely positive. Moreover, even working with finite-dimensional C^*-algebras would not actually help. In this case $\mathcal{S}$ would of course be an injective operator system.
Yet there is no evidence that the projection P obtained in Proposition 3.1, and thus naturally associated with Φ, might have anything to do with the abstract projection provided by applying the definition of injectivity.
However, we do know that $\mathrm{Ran}(P)$ is already a C^*-algebra with respect to the given product in $\mathfrak{A}$ when Φ is an endomorphism whose spectrum consists only of 0 and peripheral spectrum.

Proposition 3.3. *Let $\mathfrak{A}$ be a C^*-algebra, and Φ be an endomorphism of $\mathfrak{A}$ that is not injective. Moreover, suppose $\sigma(\Phi) = \sigma_{ph}(\Phi) \cup \{0\}$, and define $Q := \frac{1}{2\pi i} \oint_\gamma R(z, \Phi)dz$ and $P := I - Q$. Then we have:*

$$Ran\Phi = RanP \simeq \mathfrak{A}/Ker\Phi. \tag{3.8}$$

So, $RanP$ is a C^*-algebra and the restriction of Φ to $RanP$ acts as a *-automorphism.

Proof. Taking into account that Q is the projection onto $Ker\Phi$, it's enough to show that $RanP = Ran\Phi$. To this aim, take $x \in \mathfrak{A}$, then

$$x = Qx + Px \Rightarrow \Phi(x) = \Phi(Qx) + \Phi(Px) = \Phi(Px),$$

which means $RanP \subseteq Ran\Phi$. To prove the reverse inclusion, we need only recall that ϕ commutes with its resolvent, hence:

$$\Phi(x) = \Phi(Px) = P\Phi(x)$$

which finally yields the desired result as

$$RanP = Ran\Phi \simeq \mathfrak{A}/Ker\Phi. \tag{3.9}$$

To show injectivity, one must show that taking $x \in RanP$ with $\Phi(x) = 0$, it implies $x = 0$. But of course it is $\Phi(x) = 0 \Leftrightarrow x = Qx$, and therefore

$$x = Qx + Px \Longrightarrow Px = 0.$$

and so $x = 0$ since $x \in RanP$. The surjectivity follows again from the same decomposition as for any $x \in RanP$.

$$x = Px + Qx \Longrightarrow \Phi(x) = \Phi P(x) + \Phi Q(x) = \Phi P(x),$$

and then $\Phi(\mathfrak{A}) = \Phi P(\mathfrak{A})$. □

In section 2.6 of the previous chapter we came to a complete description of the spectrum of a general *-endomorphism of a finite-dimensional C^*-algebra. In particular, we saw that it is always peripheral and the worst that can happen is that 0 is also an eigenvalue. Accordingly, the hypotheses made in Proposition 3.3 are fulfilled for all finite dimensional C^*-algebras. The resulting situation can then be summed up in the following corollary, which at this point can be safely stated without proof.

Proposition 3.4. *Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra. For any *- endomorphism Φ of $\mathfrak{A}$, we have a decomposition $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}'$, where $\mathfrak{A}_0$ is the (possibly zero) kernel of Φ, and $\mathfrak{A}'$ is the range of Φ, on which Φ acts as a periodic automorphism, that is there exists $n_0 \in \mathbb{N}$ such that $\Phi^{n_0}(a) = a$ for every $a \in \mathfrak{A}'$.*

Final remarks and outlook

The main contributions of this thesis are:

- Theorem 2.2.1, which provides a description of the spectrum of a non-invertible isometry. Although a known result, it is not easily found in the literature.

- Theorem 3.1.1, which allows to recognise the transient part of the dynamics induced by a completely positive map Φ as long as we assume the spectrum of Φ does display a mass gap.

- Example 3.3 which is possibly the easiest, almost trivial, example where everything works as expected, that is the operator system corresponding to the persistent part is in fact a C^*-algebra on which Φ restricts as an automorphism.

- Theorem 2.6.3, which completely characterizes the spectrum of endomorphisms in the context of finite-dimensional C^*-algebras.

As mentioned more than once during the discussion, several questions do remain open, in particular:

- Is it possible to separate the transient part of the system from the dissipative one without assuming the presence of a mass gap? For instance assuming that the spectrum of the map Φ is the whole closed unit disk?

- Is it possible to arrive at a thorough description of the spectrum of *-endomorphisms, or even completely positive maps, in the more general context of non-commutative C^*-algebras similarly to what happens for commutative C^*-algebras?

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